PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Tides in Oceans Bounded by Meridians. I. Ocean Bounded by Complete Meridian: General Equations. II. Ocean Bounded by Complete Meridian: Diurnal Tides. with an Appendix on Fourier Expansions of Associated Legendre Functions

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IX—Tides in Oceans Bounded by Meridians

I—Ocean Bounded by Complete Meridian: General Equations

By J. PROUDMAN, F.R.S.

II—Ocean Bounded by Complete Meridian: Diurnal Tides

By A. T. DOODSON, F.R.S.

With an Appendix on Fourier Expansions of Associated Legendre Functions

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I—Ocean Bounded by Complete Meridian: General Equations

By J. PROUDMAN, F.R.S.

1—INTRODUCTION

This is the first part of a series of papers by A. T. DOODSON and myself, in which we intend to publish certain investigations which we have been carrying out intermittently for some years.

In 1916 I published an account* of a general method of treating the dynamical equations of the tides in which the ordinary differential equations were transformed into an infinite sequence of algebraic equations. One of the chief features of the treatment is that an attempt was made to deal rigorously with questions of convergence. At that time the determination of the tides in a flat rectangular sea, a flat sectorial sea, and an ocean bounded by two meridians constituted mathematical problems which were completely unsolved, and I pointed out that for basins of these shapes and of uniform depth the coefficients in my algebraic equations could easily be evaluated. It is a disadvantage of the method, however,

* 'Proc. Lond. Math. Soc.,' vol. 18, p. 1. (1916).

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as applied to these systems, that the algebraic equations are naturally arranged in a double sequence and not in a single sequence.

In 1920 G. I. TAYLOR published * his solution of the problem of the rectangular basin, and in this solution there is also an unlimited number of algebraic equations, but they are naturally arranged in a single sequence.

It then appeared advisable to examine whether methods somewhat similar to those of TAYLOR could be applied to the other problems, and with this object I considered the case of a flat semicircular sea and that of a hemispherical ocean bounded by a complete meridian, both basins having uniform depth. For the semicircular basin the attempt was completely successful,† but for the hemispherical basin it was first necessary to evaluate a number of functions analogous to BESSEL's functions. The investigation was taken over by DOODSON; he tabulated the functions‡ and then proceeded to use them. It was one of the features of the method that the unknowns of the algebraic equations had to be determined by the vanishing along the equator of a function of longitude represented by a trigonometrical series. DOODSON soon found that this determination was impracticable and in an examination of the case in which the rotation of the earth is neglected I showed§ that the corresponding series do not converge on the equator. It may be remarked that the difficulty would not arise for an ocean bounded by the meridians and a parallel of latitude other than the equator; for such a basin the tides could be determined by the method of the semicircular sea in which BESSEL's functions were replaced by DOODSON's functions.

DOODSON then developed two new methods. In one of these methods the fundamental differential equations are replaced by equations of finite differences; in the other, series are arranged in powers of the difference of longitude between the bounding meridians. It is his intention to publish his results in these papers.

In 1927 GOLDSBROUGH began to publish‖ his solutions of problems of the type in question. In his determination of forced tides there is an infinite sequence of algebraic equations, and as in TAYLOR's solution, they are naturally arranged in a single sequence. But any method is so complicated and there are so many special cases for which it is desirable to have numerical results that it appeared advisable to continue our investigations. It may be remarked that in two recent papers¶ GOLDSBROUGH has used doubly infinite sequences of equations to discuss free tidal oscillations, and that these equations are practically the same as those resulting from my method.

In the present part, I follow the general ideas of the paper of 1916, but introduce two important changes. The coordinates here used are not latitude and longitude,

* *Ibid.*, vol. 20, p. 148 (1920).

† 'Mon. Not. R. Astr. Soc., Geophys. Suppl.,' vol. 2, p. 22 (1928).

‡ *Ibid.*, vol. 1, p. 541 (1927).

§ *Ibid.*, vol. 2, p. 209 (1929).

‖ 'Proc. Roy. Soc.,' A, vol. 117, p. 692 (1928); vol. 122, p. 228 (1929); vol. 126, p. 1 (1930).

¶ *Ibid.*, vol. 132, p. 689 (1931); vol. 140, p. 241 (1933).

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but their pole is taken on the equator. Also, all subsidiary functions and constants have now zero dimensions and so are purely numerical. It is therefore necessary to start from the fundamental differential equations, but questions of expansion and of convergence remain as before, and for their treatment reference must be made to the earlier paper.

The equations are particularized for an ocean of uniform depth bounded by a complete meridian, and the tables give the numerical values of 716 coefficients of these equations. In the computation of these coefficients much assistance has been rendered by the staff of the Liverpool Observatory and Tidal Institute. The investigation is carried as far as is practicable without assigning particular values to the depth of the ocean or to the period of the tides.

2—Fundamental Equations

We shall denote by a the radius of the earth, by g the acceleration of gravity at the earth's surface, by Ω the angular speed of the earth's rotation, and by h the depth of the ocean, supposed uniform. Let O and A be two fixed points on the equator so that A is to the east of O, and let P be any variable point of the ocean. Then we shall denote by θ, χ the side OP and the angle AOP respectively of the spherical triangle AOP. Further, we shall denote by ζ the elevation of the free surface of the ocean at any time at P; by u, v the components of the current at any time at P in the directions of OP and a right angle in advance of OP respectively; and by $\bar{\zeta}$ the "equilibrium-form" of ζ corresponding to the disturbing forces.

Then the equation of continuity* may be written in the form

$$\frac{1}{a \sin\theta}\left\{\frac{\partial}{\partial\theta}(hu\sin\theta) + \frac{\partial}{\partial\chi}(hv)\right\} + \dot{\zeta} = 0, \quad \ldots \ldots \quad (2.1)$$

and the dynamical equations* as

$$\left.\begin{aligned} \dot{u} - 2\omega v &= -\frac{g}{a}\frac{\partial}{\partial\theta}(\zeta - \bar{\zeta}) \\ \dot{v} + 2\omega u &= -\frac{g}{a\sin\theta}\frac{\partial}{\partial\chi}(\zeta - \bar{\zeta}) \end{aligned}\right\}, \quad \ldots \ldots \quad (2.2)$$

where dots denote time-differentiation and

$$\omega = \Omega \sin\theta \sin\chi \quad \ldots \ldots \ldots \ldots \quad (2.3)$$

the component of the earth's angular velocity along the vertical at the point P.

We shall work with complex harmonic motion and omit the exponential time-factor.

* Lamb, "Hydrodynamics," art. 214.

Let θ' denote co-latitude and χ' east-longitude measured from the central meridian. Then for semidiurnal, diurnal and long period constituents we have* respectively

$$\left.\begin{aligned}\bar{\zeta} &= \mathrm{H}\sin^2\theta' e^{2i\chi'}\\ \bar{\zeta} &= \mathrm{H}\sin 2\theta' e^{i\chi'}\\ \bar{\zeta} &= \mathrm{H}\left(\tfrac{1}{3}-\cos^2\theta'\right)\end{aligned}\right\}, \quad \ldots \quad (2.4)$$

where H is a constant appropriate to each constituent.

From spherical trigonometry we have

$$\cos\theta' = \sin\theta\sin\chi,$$

$$\cos\theta = \sin\theta'\cos\chi',$$

$$\sin\theta'\sin\chi' = \sin\theta\cos\chi,$$

and then on substituting into (2.4) we obtain respectively

$$\left.\begin{aligned}\bar{\zeta} &= \mathrm{H}\left(\mathrm{P}_2 + \tfrac{2}{3}i\mathrm{P}_2^1\cos\chi - \tfrac{1}{6}\mathrm{P}_2^2\cos 2\chi\right)\\ \bar{\zeta} &= \mathrm{H}\left(\tfrac{2}{3}\mathrm{P}_2^1\sin\chi + \tfrac{1}{3}i\mathrm{P}_2^2\sin 2\chi\right)\\ \bar{\zeta} &= \mathrm{H}\left(\tfrac{1}{3}\mathrm{P}_2 + \tfrac{1}{6}\mathrm{P}_2^2\cos 2\chi\right)\end{aligned}\right\}, \quad \ldots \quad (2.41)$$

where P_r^n denotes FERRER's form of the Associated Legendre Function of argument $\cos\theta$.

Now it is known† that we may take two functions of position and time, ϕ and ψ, satisfying the conditions

$$\frac{\partial\phi}{\partial n} = 0, \qquad \psi = 0, \quad \ldots \quad (2.5)$$

at the coastline, where $\partial/\partial n$ denotes differentiation along a normal to the coastline, and such that

$$\left.\begin{aligned}\frac{u}{a} &= -\frac{\partial\dot{\phi}}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\dot{\psi}}{\partial\chi}\\ \frac{v}{a} &= -\frac{1}{\sin\theta}\frac{\partial\dot{\phi}}{\partial\chi} + \frac{\partial\dot{\psi}}{\partial\theta}\end{aligned}\right\}, \quad \ldots \quad (2.6)$$

over the ocean. We notice from (2.6) that ϕ and ψ are of zero dimensions. It follows from (2.1) that

$$\frac{\zeta}{h} = \frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\phi}{\partial\chi^2}\right\}, \quad \ldots \quad (2.7)$$

over the ocean.

* LAMB, "Hydrodynamics." Appendix to chap. VIII.

† PROUDMAN, 'Proc. Lond. Math. Soc.,' vol. 18, p. 1 (1916).

3—Transformation of Equations

We now take two sequences of functions of position ϕ_r, ψ_r, and two sequences of corresponding constants λ_r, μ_r, all independent of time and of zero dimensions. They are defined so as to satisfy the differential equations

$$\left.\begin{aligned}\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\phi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\,\frac{\partial^2\phi_r}{\partial\chi^2}\right\}+\lambda_r\phi_r=0\\ \frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\,\frac{\partial^2\psi_r}{\partial\chi^2}\right\}+\mu_r\psi_r=0\end{aligned}\right\}, \quad . \quad . \quad . \quad (3.1)$$

over the ocean, with the conditions

$$\frac{\partial\phi_r}{\partial n}=0, \qquad \psi_r=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

at the coastline. The functions ϕ_r and ψ_r are each subject to arbitrary constant factors, but we may deduce* that

$$\begin{aligned}\iint\left(\frac{\partial\phi_r}{\partial\theta}\frac{\partial\phi_s}{\partial\theta}+\frac{1}{\sin^2\theta}\,\frac{\partial\phi_r}{\partial\chi}\,\frac{\partial\phi_s}{\partial\chi}\right)\sin\theta\,d\theta\,d\chi&\\ =\lambda_r\iint\phi_r\phi_s\sin\theta\,d\theta\,d\chi&=\lambda_s\iint\phi_r\phi_s\sin\theta\,d\theta\,d\chi\\ =0 \qquad (s\neq r)&\\ =L_r^2 \qquad (\text{say, } s=r)\,;&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)\end{aligned}$$

$$\begin{aligned}\iint\left(\frac{\partial\psi_r}{\partial\theta}\frac{\partial\psi_s}{\partial\theta}+\frac{1}{\sin^2\theta}\,\frac{\partial\psi_r}{\partial\chi}\,\frac{\partial\psi_s}{\partial\chi}\right)\sin\theta\,d\theta\,d\chi&\\ =\mu_r\iint\psi_r\psi_s\sin\theta\,d\theta\,d\chi&=\mu_s\iint\psi_r\psi_s\sin\theta\,d\theta\,d\chi\\ =0 \qquad (s\neq r)&\\ =M_r^2 \qquad (\text{say, } s=r).&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)\end{aligned}$$

In (3.3) and (3.4) and in all the double integrals that follow, the integrations are to be taken over the whole area of the ocean. The quantities L_r and M_r are, of course, independent both of position and of time, while the arbitrary factors disappear from the ratios ϕ_r/L_r and ψ_r/M_r.

Let us arrange the sequences (λ_r) and (μ_r) in non-decreasing order of magnitude with $r = 1, 2, 3, \ldots$ and use the expansions

$$\phi=\sum_{s=1}^{\infty}\frac{p_s}{L_s}\,\phi_s, \qquad \psi=\sum_{s=1}^{\infty}\frac{p_{-s}}{M_s}\,\psi_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

where p_s and p_{-s} are numerical functions of time but not of position. Corresponding to any definite tidal motion the series in (3.5) are known to be absolutely and

* Proudman, 'Proc. Lond. Math. Soc.,' vol. 18, p. 1 (1916).

uniformly convergent over the whole ocean. On substituting into (2.7) and using (3.1) we have

$$\frac{\zeta}{h} = -\sum_{s=1}^{\infty} \frac{\lambda_s p_s}{L_s} \phi_s, \qquad (3.51)$$

but this series cannot converge nearly so rapidly as the first of (3.5).

It is now our object to derive equations in which the only variables are functions of time. For this purpose we construct the integral

$$\iint \left\{ \left[-\frac{\partial \ddot{\phi}}{\partial \theta} - \frac{1}{\sin\theta} \frac{\partial \ddot{\psi}}{\partial \chi} - 2\Omega \sin\theta \sin\chi \left(-\frac{1}{\sin\theta} \frac{\partial \dot{\phi}}{\partial \chi} + \frac{\partial \dot{\psi}}{\partial \theta} \right) + g \frac{\partial}{\partial \theta} (\zeta - \bar{\zeta}) \right]^2 + \left[-\frac{1}{\sin\theta} \frac{\partial \ddot{\phi}}{\partial \chi} + \frac{\partial \ddot{\psi}}{\partial \theta} + 2\Omega \sin\theta \sin\chi \left(-\frac{\partial \dot{\phi}}{\partial \theta} - \frac{1}{\sin\theta} \frac{\partial \dot{\psi}}{\partial \chi} \right) + \frac{g}{\sin\theta} \frac{\partial}{\partial \chi} (\zeta - \bar{\zeta}) \right]^2 \right\} \sin\theta \, d\theta \, d\chi. \qquad (3.6)$$

which is suggested by the equations of motion (2.2) after substituting from (2.6). Now substitute the finite series

$$\phi = \sum_{s=1}^{N} \frac{p_s}{L_s} \phi_s, \qquad \psi = \sum_{s=1}^{N'} \frac{p_{-s}}{M_s} \psi_s$$

into (3.6) and proceed to choose

$$\ddot{p}_1, \ddot{p}_2 \; \ldots \; \ddot{p}_N \, ; \; \ddot{p}_{-1}, \ddot{p}_{-2} \; \ldots \; \ddot{p}_{-N'}$$

so as to make the integral (3.6) a minimum for given values of

$$p_1, p_2 \; \ldots \; p_N \, ;$$

$$\dot{p}_1, \dot{p}_2, \; \ldots \; \dot{p}_N \, ; \; \dot{p}_{-1}, \dot{p}_{-2}, \; \ldots \; \dot{p}_{-N'}.$$

We thus obtain the equations

$$\ddot{p}_r + 2\Omega \sum_{s=-N'}^{N} \beta_{r,s} \dot{p}_s + \frac{gh\lambda_r}{a^2} (p_r - \Pi_r) = 0, \qquad (3.7)$$

where

$$\Pi_r = -\frac{1}{hL_r} \iint \bar{\zeta} \phi_r \sin\theta \, d\theta \, d\chi, \qquad (3.71)$$

and

$$\beta_{r,s} = -\frac{1}{L_r L_s} \iint \left(\frac{\partial \phi_r}{\partial \theta} \frac{\partial \phi_s}{\partial \chi} - \frac{\partial \phi_r}{\partial \chi} \frac{\partial \phi_s}{\partial \theta} \right) \sin\theta \sin\chi \, d\theta \, d\chi, \qquad (3.72)$$

$$\beta_{r,-s} = \frac{1}{L_r M_s} \iint \left(\sin^2\theta \frac{\partial \phi_r}{\partial \theta} \frac{\partial \psi_s}{\partial \theta} + \frac{\partial \phi_r}{\partial \chi} \frac{\partial \psi_s}{\partial \chi} \right) \sin\theta \, d\theta \, d\chi, \qquad (3.73)$$

for positive values of r and s ; also

$$\ddot{p}_{-r} + 2\Omega \sum_{s=-N'}^{N} \beta_{-r,s}\, \dot{p}_s = 0, \quad \ldots\ldots\ldots\ldots \quad (3.8)$$

where

$$\beta_{-r,s} = -\frac{1}{M_r L_s} \iint \left(\sin^2\theta \frac{\partial\psi_r}{\partial\theta}\frac{\partial\phi_s}{\partial\theta} + \frac{\partial\psi_r}{\partial\chi}\frac{\partial\phi_s}{\partial\chi} \right) \sin\chi\, d\theta\, d\chi, \quad \ldots \quad (3.81)$$

$$\beta_{-r,-s} = -\frac{1}{M_r M_s} \iint \left(\frac{\partial\psi_r}{\partial\theta}\frac{\partial\psi_s}{\partial\chi} - \frac{\partial\psi_r}{\partial\chi}\frac{\partial\psi_s}{\partial\theta} \right) \sin\theta \sin\chi\, d\theta\, d\chi, \quad \ldots \quad (3.82)$$

for positive values of r and s. The coefficients

$$\beta_{r,s}, \qquad \beta_{r,-s}, \qquad \beta_{-r,s}, \qquad \beta_{-r,-s}$$

are numerical constants, while the quantities Π_r are numerical functions of time. The equations (3.7) and (3.8) are of the type of Lagrangian equations of motion, while the quantities Π_r are Lagrangian components of the disturbing forces.

For a complex harmonic constituent of period $2\pi/\sigma$ we suppose that the time enters only through an exponential factor, and then from (3.7) and (3.8) we have

$$\left(1 - \frac{f^2\beta}{\lambda_r}\right) p_r + \frac{if\beta}{\lambda_r} \sum_{s=1}^{N} \beta_{r,s}\, p_s + \frac{if\beta}{\lambda_r} \sum_{t=1}^{N'} \beta_{r,-t}\, p_{-t} = \Pi_r \quad \ldots\ldots \quad (3.91)$$

$$if\, p_{-r} + \sum_{s=1}^{N} \beta_{-r,s} p_s + \sum_{t=1}^{N'} \beta_{-r,-t}\, p_{-t} = 0 \quad \ldots\ldots\ldots \quad (3.92)$$

where

$$f = \frac{\sigma}{2\Omega}, \qquad \beta = \frac{4\Omega^2 a^2}{gh} \quad \ldots\ldots\ldots\ldots \quad (3.93)$$

and r, s, t, are positive integers. The constants f and β are of zero dimensions.

Corresponding to any definite tidal motion the series in (3.91) and (3.92) are known to be absolutely convergent as N, N′ → ∞.

4—Ocean Bounded by a Complete Meridian

For the hemispherical ocean bounded by the complete meridian $\theta = \frac{1}{2}\pi$, the functions ϕ_r and ψ_r satisfying the equations (3.1) may each be either

$$P_r^n(\cos\theta)\cos n\chi \quad \text{or} \quad P_r^n(\cos\theta)\sin n\chi$$

where $P_r^n(\quad)$ denotes an Associated Legendre Function (Ferrer's Form), and $r = 1, 2, 3, \ldots$; while $n = 0, 1, 2, \ldots r$, in those functions containing $\cos n\chi$ and $n = 1, 2, \ldots r$, in those functions containing $\sin n\chi$. In virtue of the boundary conditions (3.2), $r + n$ is even in ϕ_r and odd in ψ_r.

It is therefore clear that ϕ_r, ψ_r and all the quantities

$$p_r,\ p_{-r},\ \Pi_r,\ \lambda_r,\ \mu_r,\ L_r,\ M_r,\ \beta_{r,s},$$

must be replaced by ϕ_r^n, ψ_r^n and

$$p_r^n,\ p_{-r}^n,\ \Pi_r^n,\ \lambda_r^n,\ \mu_r^n,\ L_r^n,\ M_r^n,\ \beta_{r,s}^{n,m},$$

respectively. We then find that

$$\lambda_r^n = \mu_r^n = r\,(r+1) \quad \ldots \quad (4.1)$$

$$L_r^n = M_r^n = \left\{\pi\,\frac{r\,(r+1)}{2r+1}\,\frac{(r+n)\,!}{(r-n)\,!}\right\}^{\frac{1}{2}} \quad (n>0), \quad \ldots \quad (4.2)$$

$$L_r^{\circ} = M_r^{\circ} = \left\{2\pi\,\frac{r\,(r+1)}{2r+1}\right\}^{\frac{1}{2}}. \quad \ldots \quad (4.21)$$

For the evaluation of the coefficients $\beta_{r,s}^{n,m}$ from (3.72), (3.73), (3.81), (3.82) we notice that

$$\int_0^{2\pi} \cos m\chi\,\cos n\chi\,\sin\chi\,d\chi = 0,$$

$$\int_0^{2\pi} \sin m\chi\,\sin n\chi\,\sin\chi\,d\chi = 0,$$

and

$$\int_0^{2\pi} \cos m\chi\,\sin n\chi\,\sin\chi\,d\chi = 0, \qquad (m \neq n \pm 1).$$

Further, we notice that

$$\int_0^{2\pi} \cos n\chi\,\sin\,(n+1)\,\chi\,\sin\chi\,d\chi = \pi \quad (n=0), \quad = \tfrac{1}{2}\pi \quad (n>0),$$

while

$$\int_0^{2\pi} \sin n\chi\,\cos\,(n+1)\,\chi\,\sin\chi\,d\chi = 0 \quad (n=0), \quad = -\tfrac{1}{2}\pi \quad (n>0).$$

We are thus led to two integrals which we shall call $\alpha_{r,s}^n$ and $\gamma_{r,s}^n$, viz.*

$$\alpha_{r,s}^n = \frac{\frac{1}{2}\pi}{L_r^n L_s^{n+1}} \int_0^{\frac{1}{2}\pi} \left\{nP_r^n\,\frac{dP_s^{n+1}}{d\theta} + (n+1)\,\frac{dP_r^n}{d\theta}\,P_s^{n+1}\right\}\sin\theta\,d\theta, \quad \ldots \quad (4.31)$$

for odd values of $r+s$, and

$$\gamma_{r,s}^n = \frac{\frac{1}{2}\pi}{L_r^n M_s^{n+1}} \int_0^{\frac{1}{2}\pi} \left\{n\,(n+1)\,P_r^n P_s^{n+1} + \sin^2\theta\,\frac{dP_r^n}{d\theta}\,\frac{dP_s^{n+1}}{d\theta}\right\} d\theta, \quad . \quad (4.32)$$

for even values of $r+s$, except that when $n=0$, these expressions must be doubled.

* When the argument of P_r^n ( ) is the variable $\cos\theta$ or x we shall omit it.

It will be convenient to divide any solution of the tidal equations into two parts, in the first of which ζ is an even function of χ and in the second of which ζ is an odd function of χ. It will also be convenient to refer to these respectively as the even and odd parts of the solution. Then we deduce that in the even part of the solution ϕ_r^n will occur only in the form $P_r^n \cos n\chi$ and ψ_r^n only in the form $P_r^n \sin n\chi$, while in the odd part of the solution ϕ_r^n will occur only in the form $P_r^n \sin n\chi$ and ψ_r^n only in the form $P_r^n \cos n\chi$.

On using only positive values of r and s, as in (3.72), (3.73), (3.81) and (3.82), we now have

$$\beta_{r,s}^{\ n,n+1} = \beta_{-r,-s}^{\ n,n+1} = \alpha_{r,s}^{\ n}, \qquad \beta_{r,s}^{\ n+1,n} = \beta_{-r,-s}^{\ n+1,n} = -\alpha_{s,r}^{\ n}, \quad . \quad . \quad (4.41)$$

$$\beta_{r,-s}^{\ n,n+1} = \beta_{-r,s}^{\ n,n+1} = \pm \gamma_{r,s}^{\ n}, \qquad \beta_{r,-s}^{\ n+1,n} = \beta_{-r,s}^{\ n+1,n} = \mp \gamma_{s,r}^{\ n}, \quad . \quad (4.42)$$

where the upper signs apply to the even part of the solution and the lower signs to the odd part of the solution. But in the even part of the solution we must take

$$\beta_{-r,-s}^{\ 0,1} = \beta_{-r,-s}^{\ 1,0} = \beta_{-r,s}^{\ 0,1} = \beta_{r,-s}^{\ 1,0} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4.43)$$

instead of (4.41) and (4.42), while in the odd part of the solution we must take

$$\beta_{r,s}^{\ 0,1} = \beta_{r,s}^{\ 1,0} = \beta_{r,-s}^{\ 0,1} = \beta_{-r,s}^{\ 1,0} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.44)$$

instead of (4.41) and (4.42).

From (3.71) we have

$$\Pi_r^n = -\frac{1}{hL_r^n}\iint \bar{\zeta}\phi_r^n \sin\theta \, d\theta \, d\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

and on substituting from (2.41) and from the present section, we have

$$\Pi_2^0 = -\sqrt{\frac{\pi}{15}}\frac{H}{h}, \qquad \Pi_2^1 = i\sqrt{\pi}\, Q_r \frac{H}{h}, \qquad \Pi_2^2 = \tfrac{1}{3}\sqrt{\frac{\pi}{5}}\frac{H}{h}, \quad . \quad (4.51)$$

for the semidiurnal constituents,

$$\Pi_r^1 = \sqrt{\pi}\, Q_r \frac{H}{h}, \qquad \Pi_2^2 = -\tfrac{2}{3}\, i\sqrt{\frac{\pi}{5}}\frac{H}{h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.52)$$

for the diurnal constituents, and

$$\Pi_2^0 = -\tfrac{1}{3}\sqrt{\frac{\pi}{15}}\frac{H}{h}, \qquad \Pi_2^2 = -\tfrac{1}{3}\sqrt{\frac{\pi}{5}}\frac{H}{h}, \quad . \quad . \quad . \quad . \quad (4.53)$$

for the long period constituents. Here

$$Q_r = \frac{2\,(-\tfrac{1}{2})^{\frac{1}{2}(r-1)}}{\{\tfrac{1}{2}\,(r-1)\}!}\,\frac{1, 3, \ldots r}{(r-2)\,(r)\,(r+1)\,(r+3)}\sqrt{(2r+1)}, \quad . \quad . \quad (4.54)$$

for odd values of r; and all other values of Π^n_r are zero.

5—Formulae in Associated Legendre Functions

In this section and the next we shall use the relationships

$$P_r^n = (1-x^2)^{\frac{1}{2}n} \frac{d^nP_r}{dx^n}, \quad \ldots \quad (5.11)$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{dP_r^n}{d\theta}\right) + \left\{r(r+1)\sin\theta - \frac{n^2}{\sin\theta}\right\} P_r^n = 0, \quad \ldots \quad (5.12)$$

$$\frac{d}{dx}\left\{(1-x^2)^n \frac{d^nP_r}{dx^n}\right\} + (r-n+1)(r+n)(1-x^2)^{n-1}\frac{d^{n-1}P_r}{dx^{n-1}} = 0, \quad . \quad (5.13)$$

$$\sin\theta \frac{dP_r^n}{d\theta} = n\cos\theta\, P_r^n - \sin\theta\, P_r^{n+1}, \quad \ldots \quad (5.14)$$

$$(2r+1)\sin\theta\, P_r^{n-1} = P_{r+1}^n - P_{r-1}^n, \quad \ldots \quad (5.15)$$

$$(2r+1)\cos\theta\, P_r^n = (r-n+1)\, P_{r+1}^n + (r+n)\, P_{r-1}^n, \quad \ldots \quad (5.16)$$

$$P_r^n(0) = (-\tfrac{1}{2})^{\frac{1}{2}(r-n)} \frac{1, 3, \ldots (r+n-1)}{\{\frac{1}{2}(r-n)\}!} \quad (r+n \text{ even}). \quad . \quad . \quad (5.17)$$

We now proceed to deduce formulae for the evaluation of the integrals

$$\int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\, P_r^nP_s^n\, d\theta, \qquad \int_0^{\frac{1}{2}\pi} \sin^2\theta\, P_r^{n-1}P_s^n\, d\theta,$$

and

$$\int_0^{\frac{1}{2}\pi} \sin\theta\, P_r^nP_s^n\, d\theta,$$

which, as will be seen in the next section, arise in the evaluation of $\alpha_{r,s}^n$ and $\gamma_{r,s}^n$.

On using (5.16) we have

$$\int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\, P_r^nP_s^n\, d\theta$$
$$= \frac{r-n+1}{2r+1}\int_0^{\frac{1}{2}\pi} \sin\theta\, P_{r+1}^nP_s^n\, d\theta + \frac{r+n}{2r+1}\int_0^{\frac{1}{2}\pi}\sin\theta\, P_{r-1}^nP_s^n\, d\theta, \quad . \quad . \quad (5.2)$$

and on using (5.15) we have

$$\int_0^{\frac{1}{2}\pi} \sin^2\theta\, P_r^{n-1}P_s^n\, d\theta = \frac{1}{2r+1}\int_0^{\frac{1}{2}\pi}\sin\theta\, P_{r+1}^nP_s^n\, d\theta - \frac{1}{2r+1}\int_0^{\frac{1}{2}\pi}\sin\theta\, P_{r-1}^nP_s^n\, d\theta. \quad . \quad (5.3)$$

Now

$$\int_0^1 (1-x^2)^{n+1}\frac{d^{n+1}P_r}{dx^{n+1}}\frac{d^{n+1}P_s}{dx^{n+1}}\, dx$$
$$= \left[(1-x^2)^{n+1}\frac{d^{n+1}P_r}{dx^{n+1}}\frac{d^nP_s}{dx^n}\right]_0^1 - \int_0^1 \frac{d}{dx}\left\{(1-x^2)^{n+1}\frac{d^{n+1}P_r}{dx^{n+1}}\right\}\frac{d^nP_s}{dx}\, dx$$
$$= -P_r^{n+1}(0)\, P_s^n(0) + (r-n)(r+n+1)\int_0^1 (1-x^2)^n \frac{d^nP_r}{dx^n}\frac{d^nP_s}{dx^n}\, dx,$$

on using (5.13); hence and similarly

$$\int_0^{\frac{1}{2}\pi} \sin\theta \, P_r^{n+1} P_s^{n+1} \, d\theta$$
$$= - P_r^{n+1}(0) \, P_s^n(0) + (r-n)(r+n+1) \int_0^{\frac{1}{2}\pi} \sin\theta \, P_r^n P_s^n \, d\theta$$
$$= - P_s^{n+1}(0) \, P_r^n(0) + (s-n)(s+n+1) \int_0^{\frac{1}{2}\pi} \sin\theta \, P_s^n P_r^n \, d\theta,$$

so that

$$\int_0^{\frac{1}{2}\pi} \sin\theta \, P_r^n P_s^n \, d\theta = \frac{P_r^{n+1}(0)\, P_s^n(0) - P_r^n(0)\, P_s^{n+1}(0)}{(r-s)(r+s+1)}. \quad . \quad . \quad . \quad (5.4)$$

When $r+n$ or $s+n+1$ is even (5.4) reduces to

$$-\frac{P_r^n(0)\, P_s^{n+1}(0)}{(r-s)(r+s+1)}, \quad . \quad . \quad . \quad . \quad (5.41)$$

and when $r+n$ or $s+n+1$ is odd it reduces to

$$\frac{P_r^{n+1}(0)\, P_s^n(0)}{(r-s)(r+s+1)}. \quad . \quad . \quad . \quad . \quad (5.42)$$

We shall also require the values of the ratios

$$\frac{P_r^{n+2}(0)}{P_r^n(0)}, \qquad \frac{P_{r+1}^{n+1}(0)}{P_r^n(0)}, \qquad \frac{P_{r-1}^{n+1}(0)}{P_r^n(0)}, \qquad (r+n \text{ even})$$

and on using (5.17) we see that these are respectively

$$-(r-n)(r+n+1), \qquad r+n+1, \qquad -(r-n). \quad . \quad . \quad . \quad (5.5)$$

6—Evaluation of $\alpha_{r,s}^n$, $\gamma_{r,s}^n$

From (4.31) we deduce, with the help of (5.14) that

$$\alpha_{r,s}^n = \frac{\frac{1}{2}\pi}{L_r^n L_s^{n+1}} \left\{ n P_r^n(0)\, P_s^{n+1}(0) - \int_0^{\frac{1}{2}\pi} \sin\theta \, P_r^{n+1} P_s^{n+1} \, d\theta \right\}, \quad . \quad . \quad . \quad (6.1)$$

except that when $n=0$ this expression must be doubled. On substituting from (5.41), (5.42) and (5.5) we have

$$\alpha_{r,s}^n = \tfrac{1}{2}\pi \, \frac{P_r^n(0)}{L_r^n} \, \frac{P_s^{n+1}(0)}{L_s^{n+1}} \left\{ n + \frac{(r-n)(r+n+1)}{(r-s)(r+s+1)} \right\}, \quad . \quad . \quad . \quad (6.11)$$

when $r+n$ and $s+n+1$ are even, and

$$\alpha_{r,s}^n = \tfrac{1}{2}\pi \, \frac{P_r^{n+1}(0)}{L_r^n} \, \frac{P_s^{n+2}(0)}{L_s^{n+1}} \, \frac{1}{(r-s)(r+s+1)}, \quad . \quad . \quad . \quad (6.12)$$

when $r+n$ and $s+n+1$ are odd, except that when $n=0$ both (6.11) and (6.12) must be doubled.

On multiplying (5.12) by $\sin\theta\, P_s^{n+1}$ and integrating, we have

$$\int_0^{\frac{1}{2}\pi}\{r(r+1)\sin^2\theta-n^2\}\,P_r^nP_s^{n+1}\,d\theta$$
$$=-\int_0^{\frac{1}{2}\pi}\frac{d}{d\theta}\left(\sin\theta\frac{dP_r^n}{d\theta}\right)\sin\theta\,P_s^{n+1}\,d\theta$$
$$=P_r^{n+1}(0)\,P_s^{n+1}(0)+\int_0^{\frac{1}{2}\pi}\sin\theta\frac{dP_r^n}{d\theta}\frac{d}{d\theta}(\sin\theta\,P_s^{n+1})\,d\theta,$$

and the integral in this is equal to

$$\int_0^{\frac{1}{2}\pi}\left\{\sin\theta\cos\theta\frac{dP_r^n}{d\theta}P_s^{n+1}+\sin^2\theta\frac{dP_r^n}{d\theta}\frac{dP_s^{n+1}}{d\theta}\right\}d\theta$$
$$=\int_0^{\frac{1}{2}\pi}\left\{n\cos^2\theta\,P_r^nP_s^{n+1}-\sin\theta\cos\theta\,P_r^{n+1}P_s^{n+1}+\sin^2\theta\frac{dP_r^n}{d\theta}\frac{dP_s^{n+1}}{d\theta}\right\}d\theta.$$

It follows that

$$\int_0^{\frac{1}{2}\pi}\left\{n(n+1)\,P_r^nP_s^{n+1}+\sin^2\theta\frac{dP_r^n}{d\theta}\frac{dP_s^{n+1}}{d\theta}\right\}d\theta=-P_r^{n+1}(0)\,P_s^{n+1}(0)$$
$$+\int_0^{\frac{1}{2}\pi}\{[r(r+1)+n]\sin^2\theta\,P_r^nP_s^{n+1}+\sin\theta\cos\theta\,P_r^{n+1}P_s^{n+1}\}\,d\theta,$$

and so, from (4.32)

$$\gamma_{r,s}^n=\frac{\frac{1}{2}\pi}{L_r^nM_s^{n+1}}\Big\{-P_r^{n+1}(0)\,P_s^{n+1}(0)$$
$$+[r(r+1)+n]\int_0^{\frac{1}{2}\pi}\sin^2\theta\,P_r^nP_s^{n+1}\,d\theta+\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\,P_r^{n+1}P_s^{n+1}\,d\theta\Big\},\quad(6.2)$$

except that when $n=0$ this expression must be doubled. On substituting from (5.2) and (5.3) we have

$$\gamma_{r,s}^n=\frac{\frac{1}{2}\pi}{L_r^nM_s^{n+1}}\Big\{-P_r^{n+1}(0)\,P_s^{n+1}(0)+\frac{r(r+2)}{2r+1}\int_0^{\frac{1}{2}\pi}\sin\theta P_{r+1}^{n+1}P_s^{n+1}\,d\theta$$
$$-\frac{r^2-1}{2r+1}\int_0^{\frac{1}{2}\pi}\sin\theta\,P_{r-1}^{n+1}P_s^{n+1}\,d\theta\Big\},\quad.\;.\;.\;.\;.\;.\quad(6.21)$$

and then on substituting from (5.41), (5.42), and (5.5) we have, after some reduction,

$$\gamma_{r,s}^n=-\tfrac{1}{2}\pi\frac{P_r^n(0)}{L_r^n}\frac{P_s^{n+2}(0)}{M_s^{n+1}}\,.$$
$$.\left\{\frac{(r-1)(r+2)[r(r+1)-n]-s(s+1)[r(r+1)+n]}{(r-s-1)(r-s+1)(r+s)(r+s+2)}\right\},\quad(6.31)$$

when $r+n$ and $s+n$ are even, and

$$\gamma_{r,s}^{n} = -\tfrac{1}{2}\pi \frac{P_r^{n+1}(0)}{L_r^n} \frac{P_s^{n+1}(0)}{M_s^{n+1}} \cdot \left\{1 - \frac{(r-1)(r+2)[r(r+1)-(n+1)] - s(s+1)[r(r+1)+(n+1)]}{(r-s-1)(r-s+1)(r+s)(r+s+2)}\right\}, \tag{6.32}$$

when $r+n$ and $s+n$ are odd, except that when $n=0$ both (6.31) and (6.32) must be doubled.

The following tables give numerical values of such of the coefficients $\alpha_{r,s}^{n}$ and $\gamma_{r,s}^{n}$ as are required in the applications.

TABLE I

$\alpha_{r,s}^{n}$.

n	r	$s=1$	3	5	7	9	11
	2	−0·41926	−0·10674	0·01673	−0·00586	0·00275	−0·00151
0	4	0·17116	−0·14707	−0·07374	0·01495	−0·00606	0·00312
	6	−0·11179	0·05692	−0·08919	−0·05580	0·01282	−0·00564
	8	0·08369	−0·03729	0·03339	−0·06396	−0·04479	0·01109
	10	−0·06707	0·02822	−0·02167	0·02343	−0·04984	−0·03739
	12	0·05602	−0·02288	0·01639	−0·01507	0·01799	−0·04082
	2		0·23868	−0·12516	0·08789	−0·06820	0·05585
2	4		−0·21228	0·01781	−0·03358	0·02911	−0·02483
	6		0·12170	−0·09972	−0·00800	−0·01358	0·01424
	8		−0·08918	0·05388	−0·06488	−0·01370	−0·00606
	10		0·07092	−0·04009	0·03349	−0·04766	−0·01478
	12		−0·05902	0·03246	−0·02501	0·02369	−0·03746
	4			0·21074	−0·12194	0·09034	−0·07255
4	6			−0·17098	0·04119	−0·04450	0·03789
	8			0·11119	−0·08905	0·01011	−0·02290
	10			−0·08605	0·05505	−0·06118	−0·00053
	12			0·07077	−0·04321	0·03620	−0·04636
	6				0·18778	−0·11413	0·08727
6	8				−0·14667	0·04906	−0·04761
	10				0·10194	−0·08024	0·01899
	12				−0·08150	0·05361	−0·05703
	8					0·17044	−0·10672
8	10					−0·13033	0·05170
	12					0·09440	−0·07330
10	10						0·15698
	12						−0·11841

Table II

$\alpha_{r,s}^{\,n}$.

n	r	$s = 2$	4	6	8	10	12
	1	0·24206	−0·11482	0·07714	−0·05835	0·04699	−0·03936
1	3	−0·24650	−0·01096	−0·01964	0·01857	−0·01611	0·01399
	5	0·12553	−0·10444	−0·02462	−0·00466	0·00731	−0·00733
	7	−0·08796	0·05015	−0·06546	−0·02421	0·00000	0·00316
	9	0·06821	−0·03567	0·03008	−0·04729	−0·02199	0·00181
	11	−0·05585	0·02818	−0·02142	0·02098	−0·03687	−0·01974
	3		0·22465	−0·12498	0·09051	−0·07159	0·05941
3	5		−0·18849	0·03277	−0·04068	0·03473	−0·02967
	7		0·11643	−0·09424	0·00278	−0·01920	0·01867
	9		−0·08803	0·05506	−0·06322	−0·00614	−0·01029
	11		0·07134	−0·04224	0·03530	−0·04721	−0·00927
	5			0·19844	−0·11810	0·08904	−0·07238
5	7			−0·15747	0·04613	−0·04656	0·03970
	9			0·10634	−0·08438	0·01527	−0·02539
	11			−0·08380	0·05448	−0·05908	0·00374
	7				0·17852	−0·11031	0·08531
7	9				−0·13779	0·05077	−0·04804
	11				0·09798	−0·07657	0·02171
	9					0·16331	−0·10337
9	11					−0·12395	0·05214
11	11						0·15132

Table III

$\alpha_{r,s}^{\,n}$.

n	r	$s = 2$	4	6	8	10	12
	1	−0·24206	0·05413	−0·02439	0·01395	−0·00904	0·00634
0	3	−0·15095	−0·11392	0·03043	−0·01522	0·00932	−0·00635
	5	0·03740	−0·09032	−0·07538	0·02155	−0·01132	0·00719
	7	−0·01790	0·02502	−0·06443	−0·05641	0·01672	−0·00903
	9	0·01064	−0·01285	0·01877	−0·05008	−0·04508	0·01366
	11	−0·00709	0·00803	−0·01000	0·01501	−0·04096	−0·03755
	3		−0·06485	0·01863	−0·00954	0·00591	−0·00404
2	5		−0·05441	−0·04885	0·01429	−0·00759	0·00485
	7		0·01532	−0·04244	−0·03803	0·01139	−0·00619
	9		−0·00792	0·01245	−0·03400	−0·03093	0·00943
	11		0·00497	−0·00666	0·01023	−0·02820	−0·02601
	5			−0·03623	0·01152	−0·00630	0·00408
4	7			−0·03409	−0·03321	0·01024	−0·00564
	9			0·01027	−0·03050	−0·02857	0·00883
	11			−0·00557	0·00929	−0·02699	−0·02466
	7				−0·02386	0·00802	−0·00456
6	9				−0·02387	−0·02438	0·00778
	11				0·00750	−0·02321	−0·02239
8	9					−0·01721	0·00600
	11					−0·01789	−0·01886
10	11						−0·01317

TABLE IV

$\alpha_{r,s}^{n}$.

n	r	s = 1	3	5	7	9	11
	2		−0·09744	0·02555	−0·01243	0·00744	−0·00498
1	4		−0·07353	−0·06170	0·01737	−0·00898	0·00563
	6		0·01964	−0·05150	−0·04474	0·01312	−0·00702
	8		−0·00983	0·01472	−0·03917	−0·03502	0·01054
	10		0·00602	−0·00773	0·01161	−0·03152	−0·02874
	12		−0·00410	0·00491	−0·00627	0·00955	−0·02635
	4			−0·04712	0·01437	−0·00764	0·00485
3	6			−0·04231	−0·03981	0·01200	−0·00650
	8			0·01238	−0·03568	−0·03276	0·00998
	10			−0·00657	0·01069	−0·02981	−0·02752
	12			0·00420	−0·00580	0·00909	−0·02538
	6				−0·02897	0·00951	−0·00531
5	8				−0·02823	−0·02824	0·00887
	10				0·00870	−0·02645	−0·02517
	12				−0·00479	0·00818	−0·02354
	8					−0·02009	0·00689
7	10					−0·02052	−0·02133
	12					0·00654	−0·02057
9	10						−0·01497
	12						−0·01578

TABLE V

$\Upsilon_{r,s}^{n}$.

n	r	s = 2	4	6	8	10	12
	2	0·18042	0·24206	−0·06909	0·03535	−0·02187	0·01497
0	4	−0·33146	0·10006	0·26723	−0·08595	0·04708	−0·03052
	6	0·16356	−0·32271	0·06926	0·27720	−0·09435	0·05362
	8	−0·11524	0·15099	−0·31757	0·05296	0·28254	−0·09942
	10	0·09012	−0·10561	0·14397	−0·31431	0·04287	0·28586
	12	−0·07433	0·08277	−0·09977	0·13954	−0·31206	0·03601
	2		0·27732	−0·07967	0·04080	−0·02525	0·01729
2	4		0·19074	0·23620	−0·07257	0·03923	−0·02528
	6		−0·17676	0·14056	0·22753	−0·07393	0·04143
	8		0·09072	−0·19205	0·10961	0·22369	−0·07550
	10		−0·06471	0·09344	−0·19799	0·08952	0·22147
	12		0·05108	−0·06585	0·09302	−0·20117	0·07556
	4			0·32109	−0·10214	0·05580	−0·03614
4	6			0·21905	0·26519	−0·08480	0·04739
	8			−0·15093	0·17804	0·24981	−0·08219
	10			0·07888	−0·17089	0·14712	0·24187
	12			−0·05623	0·08514	−0·18036	0·12480
	6				0·34140	−0·11483	0·06526
6	8				0·23421	0·28189	−0·09253
	10				−0·13617	0·20130	0·26436
	12				0·07192	−0·15706	0·17260
	8					0·35324	−0·12308
8	10					0·24357	0·29280
	12					−0·12660	0·21711
10	10						0·36101
	12						0·24989

TABLE VI

$\gamma_{r,s}{}^{n}$.

n	r	$s = 1$	3	5	7	9	11
	1		0·22643	−0·05937	0·02888	−0·01729	0·01156
1	3		0·16573	0·21161	−0·06312	0·03333	−0·02110
	5		−0·19872	0·11179	0·21125	−0·06822	0·03782
	7		0·10047	−0·20743	0·08335	0·21144	−0·07154
	9		−0·07144	0·09926	−0·20980	0·06626	0·21162
	11		0·05625	−0·06978	0·09701	−0·21077	0·05494
	3			0·30418	−0·09277	0·04929	−0·03129
3	5			0·20737	0·25296	−0·07948	0·04375
	7			−0·16180	0·16177	0·23998	−0·07848
	9			0·08391	−0·18024	0·13034	0·23363
	11			−0·05988	0·08885	−0·18836	0·10871
	5				0·33279	−0·10924	0·06100
5	7				0·22764	0·27452	−0·08905
	9				−0·14267	0·19090	0·25777
	11				0·07500	−0·16332	0·16097
	7					0·34800	−0·11935
7	9					0·23939	0·28786
	11					−0·13093	0·20990
9	9						0·35749
	11						0·24701

TABLE VII

$\gamma_{r,s}{}^{n}$.

n	r	$s = 1$	3	5	7	9	11
	1	−0·18750	0·42962	−0·20196	0·13888	−0·10691	0·08720
0	3	−0·11693	−0·01116	0·36383	−0·16464	0·11396	−0·08874
	5	0·02897	−0·22126	−0·00277	0·34413	−0·15202	0·10465
	7	−0·01387	0·07148	−0·24892	−0·00108	0·33395	−0·14513
	9	0·00824	−0·03867	0·08593	−0·26208	−0·00053	0·32765
	11	−0·00549	0·02475	−0·04864	0·09362	−0·26983	−0·00030
	3		0·19535	0·29270	−0·11647	0·07746	−0·05924
2	5		−0·11708	0·11665	0·27305	−0·11099	0·07436
	7		0·04668	−0·14520	0·08506	0·25995	−0·10619
	9		−0·02679	0·05610	−0·16114	0·06706	0·25132
	11		0·01761	−0·03288	0·06220	−0·17100	0·05538
	5			0·27346	0·28895	−0·10576	0·06821
4	7			−0·13053	0·17723	0·27979	−0·10683
	9			0·06117	−0·14046	0·13682	0·26925
	11			−0·03783	0·06055	−0·15267	0·11209
	7				0·30835	0·28711	−0·10001
6	9				−0·13783	0·21263	0·28368
	11				0·07023	−0·13783	0·17093
8	9					0·32809	0·28604
	11					−0·14243	0·23578
10	11						0·34077

TABLE VIII

$\gamma_{r,s}^{n}$.

n	r	$s = 2$	4	6	8	10	12
	2	0·10417	0·29646	−0·12615	0·08537	−0·06562	0·05363
1	4	−0·10482	0·06712	0·26723	−0·11370	0·07736	−0·05988
	6	0·03569	−0·14939	0·04779	0·25305	−0·10699	0·07260
	8	−0·01921	0·05338	−0·16738	0·03693	0·24469	−0·10281
	10	0·01215	−0·03021	0·06139	−0·17729	0·03004	0·23917
	12	−0·00842	0·01988	−0·03572	0·06609	−0·18360	0·02530
	4		0·24362	0·29044	−0·11017	0·07209	−0·05478
3	6		−0·12498	0·15149	0·27696	−0·10871	0·07184
	8		0·05486	−0·14243	0·11389	0·26516	−0·10537
	10		−0·03288	0·05849	−0·15640	0·09166	0·25665
	12		0·02221	−0·03524	0·06303	−0·16592	0·07679
	6			0·29371	0·28789	−0·10251	0·06528
5	8			−0·13465	0·19699	0·28196	−0·10527
	10			0·06617	−0·13898	0·15543	0·27256
	12			−0·04193	0·06233	−0·14965	0·12933
	8				0·31942	0·28651	−0·09803
7	10				−0·14036	0·22531	0·28507
	12				0·07359	−0·13692	0·18395
9	10					0·33505	0·28565
	12					−0·14414	0·24459
11	12						0·34556

II—Ocean Bounded by Complete Meridian: Diurnal Tides

By A. T. Doodson, F.R.S.

1—Introduction

In Part I, the problem of determining the tides in an ocean bounded by a complete meridian has been reduced by Proudman to the solution of an infinite number of simultaneous equations, but if the methods of least squares are used a finite number of equations can be considered as giving an adequate representation of the solution.

This part deals with the numerical solution of the equations resulting from the use of 63 coordinates or variables, related together by six sets of equations.

The solution of so many simultaneous equations is a somewhat formidable task for even a particular value of the depth of the ocean, but a general solution was sought in which each coordinate is expressed in terms of the principal coordinates, and these again in terms of the two most important coordinates. At each stage the coefficients are series in powers of the reciprocal of the depth. By this method a resonance-equation was derived from which the critical depths for resonance could be readily obtained.

The solution (for the diurnal tide K_1) has been adequately illustrated for four depths, two of which are critical, and these suffice to illustrate the change of tide with depth, from an infinite depth to a depth smaller than the mean depth experienced terrestrially.

A very important by-product of this work is that of the tabulation of Fourier Expansions of the Associated Legendre Functions. This work is added as an Appendix to this part, where also the advantages of these expansions are explained.

2—The Associated Legendre Functions

In Part I the solution has been obtained as a series of terms involving $P_r^n(\cos\theta)/L_r^n$ where $P_r^n(\cos\theta)$ is an Associated Legendre Function, and L_r^n is a certain constant. In the Appendix to this present part we have provided tables of a function $F_r^n(\theta)$ which is more generally useful than $P_r^n(\cos\theta)/L_r^n$, with the relation

$$\frac{P_r^n(\cos\theta)}{L_r^n} = \pi_r^n F_r^n(\theta), \quad \ldots \ldots \ldots \ldots \quad (2.1)$$

where

$$\pi_r^n = \{\tfrac{1}{2}\pi r(r+1)\}^{-\frac{1}{2}} \ (n > 0), \quad \ldots \ldots \ldots \quad (2.2)$$

$$\pi_r^o = \{\pi r(r+1)\}^{-\frac{1}{2}}. \quad \ldots \ldots \ldots \ldots \quad (2.3)$$

3—Formulae for Auxiliary Functions

The solution given by Proudman makes use of two auxiliary functions, ϕ and ψ, which can be written for the diurnal case as

$$\phi = \sum_{r,n} p_r^n \pi_r^n \mathrm{F}_r^n(\theta) \sin n\chi \, e^{i\sigma t}, \quad \ldots \ldots \ldots \quad (3.1)$$

$$\psi = \sum_{r,n} p_{-r}^n \pi_r^n \mathrm{F}_r^n(\theta) \cos n\chi \, e^{i\sigma t}, \quad \ldots \ldots \ldots \quad (3.2)$$

where p_r^n and p_{-r}^n are the Lagrangian Coordinates.

4—Formulae for the Lagrangian Coordinates

The diurnal tide is taken for the case $\sigma/2\Omega = f = 0.5$, corresponding to the luni-solar diurnal tide K_1.

In order to deal with real quantities throughout, and to avoid the continual writing of $i = \sqrt{-1}$, we shall deal with ip_s^n where the coordinates are imaginary.

The equations (Part I, (3.91), (3.92)), for the diurnal case, then take the form

$$p_r^n = \Pi_r^n + \frac{\beta}{2\lambda_r}\{\tfrac{1}{2}p_r^n - \sum_{s,m} \beta_{r,s}^{n,m}\, ip_s^m - \sum_{t,m} \beta_{r,-t}^{n,m}\, ip_{-t}^m\} \quad (r, n, t, \text{odd};\ s, m \text{ even}), \quad (4.1)$$

$$ip_r^n = i\Pi_r^n + \frac{\beta}{2\lambda_r}\{\tfrac{1}{2}ip_r^n + \sum_{s,m} \beta_{r,s}^{n,m} p_s^m + \sum_{t,m} \beta_{r,-t}^{n,m} p_{-t}^m\} \quad (r, n, t \text{ even};\ s, m \text{ odd}), \quad (4.2)$$

$$p_{-r}^n = \Pi_{-r}^n + 2\sum_{s,m} \beta_{-r,-s}^{n,m}\, ip_{-s}^m \quad (r, m \text{ even};\ s, n \text{ odd}), \quad \ldots \ldots \ldots \quad (4.3)$$

$$ip_{-r}^n = i\Pi_{-r}^n - 2\sum_{s,m} \beta_{-r,-s}^{n,m} p_{-s}^m \quad (r, m \text{ odd};\ s, n \text{ even}), \quad \ldots \ldots \ldots \quad (4.4)$$

where

$$\Pi_{-r}^n = 2\sum_{s,m} \beta_{-r,s}^{n,m}\, ip_s^m \quad (r, s, m \text{ even};\ n \text{ odd}), \quad \ldots \ldots \quad (4.5)$$

$$i\Pi_{-r}^n = -2\sum_{s,m} \beta_{-r,s}^{n,m} p_s^m \quad (r, s, m \text{ odd};\ n \text{ even}), \quad \ldots \ldots \quad (4.6)$$

$$\lambda_r = r(r+1). \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (4.7)$$

The values of $\beta_{r,s}^{n,m}$, etc., are obtained from the values of α and γ tabulated in Part I, according to the relations appropriate to "the odd solution" of Part I, (4.41), (4.42), and (4.44), and these relations also determine the statements as to r, s, m, n, t, being odd or even in the above equations.

5—Transformation of One Equation

We shall transform (4.3) by substituting from (4.4), whence we obtain the form

$$p_{-r}{}^{n} = \Pi_{-r}{}^{n} + 2 \sum_{s,\,m} \beta_{-r,\,-s}{}^{n,\,m} i\Pi_{-s}{}^{m} - 4 \sum_{s,\,m} B_{-r-s}{}^{n,\,m} p_{-s}{}^{m}.$$

The coefficients of $i\Pi_{-s}{}^{m}$ and of $p_{-s}{}^{m}$ in the expansion are given in Table I, p. 305, the interpretation of which is, for example,

$$p_{-2}{}^{1} = \Pi_{-2}{}^{1} + (0.4841\, i\Pi_{-1}{}^{0} + 0.3019\, i\Pi_{-3}{}^{0} + \ldots) + (0.3746\, p_{-2}{}^{1} + 0.0214\, p_{-4}{}^{1} + \ldots).$$

This process could be continued by further substitutions, now from Table I, for the values of $p_{-s}{}^{m}$ on the right, but the convergence would be slow, as we see from the size of the coefficient of $p_{-2}{}^{1}$ in the above example. We have, however, transferred this term to the left of the equation, and then divided throughout by (1 – 0.3746), whence we got for the first equation

$$p_{-2}{}^{1} = 1.5990\, \Pi_{-2}{}^{1} + (0.7742\, i\Pi_{-1}{}^{0} + \ldots) + (0\cdot0342\, p_{-4}{}^{1} + \ldots)$$

This process was effected for the first three equations only (*i.e.*, for $n = 1$ and $r = 2$, 4, 6), after which further substitutions for $p_{-s}{}^{m}$ were made ; by successive approximations we quickly obtained

$$p_{-r}{}^{n} \text{ in terms of } \Pi_{-r}{}^{n} \text{ and } i\Pi_{-s}{}^{m}.$$

Now that the odd and even terms are automatically taken care of, we can replace the notation $\Pi_{-r}{}^{n}$ by $\Pi_{-s}{}^{m}$, so generalizing s, and obtain, as in Table IV,

$$p_{-r}{}^{n} \text{ in terms of } \Pi_{-s}{}^{m} \text{ and } i\Pi_{-s}{}^{m}. \qquad (5.1)$$

The numerical processes involved are very simply effected ; they consist of placing two columns of figures side by side, multiplying terms adjacent to one another (in the same row), and summing the products continuously on the machine.

6—Outline of Treatment of Equations

The numerical representations of equations (4.5), (4.6), (5.1) replacing (4.3), (4.4), (4.1), (4.2), are set forth in Tables II to VII respectively. As the odd and even characteristics are automatically ensured in future, we replace t by s in (4.1) and (4.2). The arrangement of the tables in this order depends upon the possibilities of solution in consecutive powers of β. To begin with, Π_r^n is known from Part I (4.52) and (4.54) as follows:—

$$n=1\begin{cases} r \;= & 1 \qquad\quad 3 \qquad\quad 5 \qquad\quad 7 \qquad\quad 9 \qquad\quad 11 \\ \Pi_r^n = & -0\cdot7675,\ -0\cdot1954,\ 0\cdot0306,\ -0\cdot0100,\ 0\cdot0050,\ -0\cdot0028 \times \mathrm{H}/h \end{cases} \tag{6.1}$$

$$n=2\begin{cases} r \;= & 2 \\ i\Pi_r^n = & 0\cdot5284\ \mathrm{H}/h \end{cases} \tag{6.2}$$

Hence, from Tables II, III, the values of Π_{-r}^n, $i\Pi_{-r}^n$ can be computed to the same zero order in β by putting $p_r^n = \Pi_r^n$. Then from Table IV the values of p_{-r}^n become known, and thence ip_{-r}^n from Table V, to the same zero order in β. These stages having been completed, Tables VI and VII can be used to give coefficients to the first order of β in p_r^n and ip_r^n. These are then used to commence a new cycle, and so on for successive cycles.

It would have been possible to have transformed (4.4) in the same way as (4.3), but simple estimation of the number of computations involved in so doing, and in the resulting processes, showed that much extra labour would be entailed by such a procedure. Similarly, the possibilities of elimination of p_{-t}^m and ip_{-t}^m from equations (4.1) and (4.2) were considered, but the advantages of the method outlined above were shown to be enormously great, seeing that in any case p_{-t}^m and ip_{-t}^m are required in the final solution.

The direct application, however, of the method outlined above, would be to give for each coordinate a power-series in β, so that resonance would only be indicated by the divergence of these series. The next simple possibility would be to take H/h and p_1^1 as independent variables, and to express all other coordinates as a linear function of these, with series-coefficients in β. In the determination of the coefficients involving H/h we should put $p_1^1 = 0$, and for those involving p_1^1 we should put $\mathrm{H}/h = 0$, and therefore $\Pi_r^n = 0$, with $\Pi_1^1 = p_1^1$ for the first stage. The equation for p_1^1 in Table VI could be ignored until the end, when it would be possible to substitute in it for all coordinates, and so get p_1^1 as a linear function of H/h and itself. This equation would then give p_1^1 as a ratio of two power-series in β. The resonant cases would thus be simply indicated by the vanishing of the denominator.

A simple examination of the coefficients likely to result showed that even this modification was not of practical value, and a systematic examination on similar lines was made, with a larger number of independent variables. The method that promised to be most economical in labour was to take each coordinate in terms of

$$\mathrm{H}/h,\ p_1^1,\ ip_2^2,\ p_3^1,\ p_3^3,\ ip_4^2,\ ip_4^4. \tag{6.3}$$

With this choice a fairly rapid convergence up to $\beta = 40$ was indicated.

It was decided, for convenience, to write

$$x = \beta/40, \tag{6.4}$$

which explains the amendment of formulae (4.1) and (4.2) under Tables VI and VII.

7—Computation of Coordinates in Terms of the Independent Coordinates

We shall take as an example the case $H/h = 1$, and all the other independent coordinates of (6.3) to be zero.

Three sheets were prepared and lettered A, B, C. On Sheet A (see Table VIII), were entered the results of computations for Π_{-r}^{n} and $i\Pi_{-r}^{n}$, Sheet B gave ip_r^n and ip_{-r}^{n}, and Sheet C gave p_r^n and p_{-r}^{n}. The columns on these sheets were headed with powers of x, and the spacing of s, m was arranged to fit appropriate tables among Tables II to VII. The upper and lower divisions of Sheets A, B, C, were denoted by (*a*) and (*b*) respectively.

The procedure is systematically set forth as :—

Compute	From	Enter on	
Π_{-r}^{n}	Table II and Ba	Aa	for the same power of x.
$i\Pi_{-r}^{n}$	Table III and Ca	Ab	
p_{-r}^{n}	Table IV and all A	Cb	
ip_{-r}^{n}	Table V and Cb and Ab	Bb	
p_r^n	Table VI and all B	Ca	for the next power of x, using $\frac{1}{2}p_r^n$, $\frac{1}{2}ip_r^n$ from the present power.
ip_r^n	Table VII and all C	Ba	

In the case H/h, we wrote dashes in B and C against the rest of the independent coordinates. In C we entered the values of Π_r^n in the first column, from (6.1), omitting $r = 1, 2, 3$. This column was placed alongside the first column of Table III, corresponding terms were multiplied together and the products summed continuously on the machine : the result is $i\Pi_1^0$ placed in the first column of Ab. Moving C along Table III a similar procedure gave $i\Pi_3^0$, and so on, until the entries of the first column of Ab were completed.

There are thus no values of $i\Pi_r^n$ to be entered in Ba so that the use of this with Table II gives zero results in Aa.

The rest of the processes are very similar, such additions as are needed in the use of Tables V, VI, and VII, being easily made. The multiples $20/\lambda_r$ required in Tables V and VI are at the feet of the tables.

Checks of the computations were effected by summation methods; by applying, for example, Ba to the horizontal sums of the coefficients in Table II, when the result equalled the sum of the values Π_{-r}^{n}. Similar elaborate checks were made throughout the work.

The results of this stage are given in Tables VIII—XIV. Only in Table VIII, for purposes of illustration of the method, has it been thought needful to give Π_{-r}^{n} and $i\Pi_{-r}^{n}$, seeing that these are only intermediate functions not required in the later stages. The interpretation of these tables is fairly obvious, but an example may help ; thus Table IX C indicates that that part of the expansion of p_5^3 depending upon p_1^1 is

$$(0{\cdot}0677\,x - 0{\cdot}0037\,x^2 + 0{\cdot}0040\,x^3 + \ldots)\,p_1^1.$$

8—Equations for "Independent Coordinates"

When using Tables VI and VII there were six equations, those for the independent coordinates, which were not used. The process of substitution of the results of §7 in these six equations is a straightforward matter. We have, for example, from Table VI,

$$p_1^1 = \Pi_1^1 + x.\frac{20}{\lambda_r}\{\tfrac{1}{2}p_1^1 + \sum_{s,m}(-\beta_{1,s}^{1,m})\, ip_s^m\}$$

or

$$(1-5x)\, p_1^1 = \Pi_1^1 + x\frac{20}{\lambda_r}\{\sum_{s,m}(-\beta_{1,s}^{1,m})\, ip_s^m\}.$$

After substituting for ip_s^m in this equation the terms on the left were transferred to the right, so collecting all terms in p_1^1.

The results are shown in Table XV, which gives six simultaneous equations, with coefficients which are power series in x. These power series are tabulated only to x^7, but it is clear that further terms can be extrapolated by multiplying successively by $-1/3$, and such terms were used in the subsequent processes.

9—Solution of Six Simultaneous Equations whose Coefficients are Power-series

The solution of the equations of Table XV is at first sight rather a formidable problem, but it is actually effected with ease, using familiar methods of procedure. The equations were treated in pairs, to eliminate one variable at a time, and this operation involved the multiplication of two series.

Suppose that $a_0+a_1\,x+a_2\,x^2+a_3\,x^3$ has to be multiplied by $b_0+b_1\,x+b_2\,x^2+b_3\,x^3$, then the coefficients of the second series can be written vertically upwards and placed alongside the coefficients of the first series, written vertically downwards. By sliding one column up and down we get the positions for multiplication for successive powers of x; the arrangements to give coefficients of 1, x, x^2, . . . are as follows :—

$$\begin{array}{ccc} b_1 & b_2 & b_3 \\ & & a_0\;\; b_2 \\ a_0\;\; b_0 & \left\{\begin{array}{l} a_0\;\; b_1 \\ a_1\;\; b_0 \end{array}\right. & \left\{\begin{array}{l} a_1\;\; b_1 \\ a_2\;\; b_0 \end{array}\right. \\ a_1 & a_2 & a_3 \end{array}$$

That is, the coefficient of x^2 is $a_0\,b_2 + a_1\,b_1 + a_2\,b_0$.

The double application of this process, where the sum of two products (each of two series) is required, is also simple.

As an example, taking equations (XV e) and (XV f), to eliminate ip_4^2 we wrote on a slip of paper the coefficients under ip_4^2, in reverse order, but changing the signs of the coefficients of (XV e). These were written vertically and placed alongside the first column. The first three terms of the product are obtained from the arrangements

0·0000	—0·0356	0·0000	—0·0284	0·0000	—0·0024
0·0490	0·0000	0·0490	—0·0356	0·0490	—0·0284
·	·	—0·0013	0·0000	—0·0013	—0·0356
·	·	·	·	—0·0026	0·0000
·	·	·	·	·	·
·	·	·	·	·	·
·	·	·	·	·	·
0·0000	0·0643	0·0000	—0·0527	0·0000	0·0041
—0·0138	1·0000	—0·0138	0·0643	—0·0138	—0·0527
		0·0264	1·0000	0·0264	0·0643
				—0·0030	1·0000

and the machine-sums of the products give

—0·0138 0·0273 —0·0020.

These results are given in the first column of Table XVI, which gives two equations resulting from the elimination of ip_4^2 and ip_4^4.

Successive applications of these principles led to the results of Tables XVII, XVIII, XIX. During the process of this work a rather curious development took place: the coefficients for x, x^3, x^5 . . . oscillated in sign, and so did those of x^2, x^4, x^6, ... An arbitrary multiplication by $(1 + x^2)$ at an appropriate stage tended to counteract this tendency. Further, it had the advantage of doubling the magnitudes of the coefficients near $x = 1$, a matter of some importance as the repeated multiplication of series tended to give small coefficients for $x = 1$.

Checks were made by comparing the results at each stage with the independent solutions of the original six equations on taking $x = 0 \cdot 5$ and $x = 1$. Small adjustments of the higher terms were occasionally made to maintain this correspondence.

The convergence is quite satisfactory; ultimately the rate is very rapid: but for the oscillatory character of the earlier coefficients leading to increasing size of coefficients, and therefore to loss of accuracy, it would be possible to proceed a stage further than Table XIX to give p_1^1 in terms of H/h. The denominator would then give the resonance equation.

10—The Resonance Equation

If we write Table XIX in the symbolic form

$$A.p_1^1 + B.ip_2^2 + C.H/h = 0,$$

$$D.p_1^1 + E.ip_2^2 + F.H/h = 0,$$

then we get the resonance cases given by

$$\mathrm{AE} - \mathrm{BD} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10\cdot 1)$$

We have been content, however, to resort to numerical expressions for these equations at intervals of 0·1 in x, and we obtain results as follows :—

x	β	AE—BD	$p^1{}_1$	$ip_2{}^2$
0·0	0	4·0000	—0·7675 H/h	0·5284 H/h
0·1	4	2·6397	—1·3396	0·6851
0·2	8	1·0456	—3·4024	1·3319
0·3	12	—0·3096	10·158	—3·028
0·4	16	—1·0765	2·2420	—0·4810
0·5	20	—1·1639	1·3393	—0·1673
0·6	24	—0·7685	1·0328	0·0003
0·7	28	—0·2510	1·0657	0·2996
0·8	32	0·0841	—0·1522	—0·8454
0·9	36	0·1550	0·3006	—0·2464
1·0	40	0·0814	0·3194	—0·1454

. (10.2)

By interpolation, and subsidiary calculations, we get resonance indicated as taking place when

$$\beta = 10\cdot 948 \text{ and } \beta = 30\cdot 63, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.3)$$

corresponding to depths of

$$h = 26{,}520 \text{ ft. and } h = 9{,}480 \text{ ft.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.4)$$

11—Computation of ϕ, ψ, u, v

From the proceeding results it appeared that the solution would be adequately illustrated by taking the cases

$$\beta = 10\cdot 948,\ 20,\ 30\cdot 63, \text{ and } 40. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.1)$$

For the two cases of resonance all coordinates are infinitely large and change sign as β changes through the resonant value ; for computation we have taken $p_1{}^1$ at the nominal value of unity.

The derivation of the values of the coordinates follows simply and systematically from Tables XIX to VIII, and the results are given for the selected cases in Tables XX(a) to XX(d).

The values of ϕ and ψ follow from (3.1) and (3.2), using the tables for $\mathrm{F}_r{}^n(\theta)$ given in the Appendix, and the results are given in Tables XXI and XXII. The values for the resonant cases, of course, are to be multiplied by $\pm\infty$.

The components of velocity, u and v, are obtained from the equations Part I (2.2), which give

$$\frac{u}{\sigma a} = -i\frac{\partial\phi}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial\psi}{\partial\chi},$$

$$\frac{v}{\sigma a} = i\frac{\partial\psi}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial\phi}{\partial\chi}.$$

The differentiation of the Fourier series, whether in θ or χ, is of course very simply effected. With regard to the operation of dividing or multiplying by $\sin\theta$, we use the formulae given below.

$$2\sin\theta\,(a_0 + a_2\cos 2\theta + a_4\cos 4\theta + \ldots) = (2a_0 - a_2)^* \sin\theta + (a_2 - a_4)\sin 3\theta + \ldots$$

$$2\sin\theta\,(a_1\cos\theta + a_3\cos 3\theta + \ldots) = (a_1 - a_3)\sin 2\theta + (a_3 - a_5)\sin 4\theta + \ldots$$

$$-2\sin\theta\,(b_1\sin\theta + b_3\sin 3\theta + \ldots) = (0 - b_1)^* + (b_1 - b_3)\cos 2\theta + \ldots$$

$$-2\sin\theta\,(b_2\sin 2\theta + b_4\sin 4\theta + \ldots) = (0 - b_2)^*\cos\theta + (b_2 - b_4)\cos 3\theta + \ldots$$

The terms marked with an asterisk (*) require special treatment; otherwise the new terms are simply derived from the old terms by taking first differences. Such a procedure can be repeated indefinitely.

Similarly we have, for instance, since $P_r^n(\cos\theta)$ has $\sin^n\theta$ as a factor if $n \geq 1$

$$-(a_0 + a_2\cos 2\theta + a_4\cos 4\theta + a_6\cos 6\theta)/2\sin\theta = (a_2 + a_4 + a_6)\sin\theta + (a_4 + a_6)\sin 3\theta + a_6\sin 5\theta$$

$$-(a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta)/2\sin\theta = (a_3 + a_5)\sin 2\theta + a_5\sin 4\theta$$

provided

$$a_0 + a_2 + a_4 + a_6 = 0 \quad (*)$$

$$a_1 + a_3 + a_5 = 0 \quad (*)$$

Also

$$(b_1\sin\theta + b_3\sin 3\theta + b_5\sin 5\theta)/2\sin\theta = \tfrac{1}{2}(b_1 + b_3 + b_5)^* + (b_3 + b_5)\cos 2\theta + b_5\cos 4\theta$$

$$(b_2\sin 2\theta + b_4\sin 4\theta + b_6\sin 6\theta)/2\sin\theta = (b_2 + b_4 + b_6)\cos\theta + (b_4 + b_6)\cos 3\theta + b_6\cos 5\theta$$

Note the terms with the asterisks.

In all these cases the process is that of continued summation of the terms beginning with the last term in the series.

As an example, take ϕ with $n = 1$, $\beta = 40$. Then $\partial\phi/\partial\chi$ is a Fourier series in $\sin s\theta$, and $\operatorname{cosec}\theta\ \partial\phi/\partial\chi$ is then a Fourier series in $\cos s\theta$; the computations give

s	$\partial\phi/\partial\chi$	s	$\dfrac{1}{2\sin\theta}\cdot\dfrac{\partial\theta}{\partial\chi}$
1 . . .	0·1520	0 . . .	$\frac{1}{2}$ [0·1357]
3 . . .	− 0·0195	2 . . .	− 0·0163
5 . . .	0·0050	4 . . .	0·0032
7 . . .	− 0·0019	6 . . .	− 0·0018
9 . . .	0·0007	8 . . .	0·0001
11 . . .	− 0·0006	10 . . .	− 0·0006

The second set we obtained from the first by repeated additions commencing at the bottom (− 6, − 6 + 7, − 6 + 7 − 19 . . .).

The values of u and v are given in Tables XXIII and XXIV.

12—Questions of Convergence and the Computation of ζ

The convergence manifested by the coordinates given in Table XX is obviously sufficient, as also is that of ϕ and ψ given in Tables XXI and XXII.

The values of ζ can now theoretically be obtained in three ways. Firstly, we have from Part I (3.51),

$$\frac{\zeta}{h} = -\sum_r \lambda_r p_r^n\, \pi_r^n F_r^n(\theta) \sin n\chi\, e^{i\sigma t} \qquad (12.1)$$

The value of λ_r is $r(r+1)$ and the supreme objection to this formula is that the convergence of the resulting series is very slow.

Secondly, we can use the second equation of Part I (2.2), where

$$\frac{\partial}{\partial\chi}\left(\frac{\zeta - \bar{\zeta}}{h}\right) = -\tfrac{1}{4}\beta\left(i\frac{v}{\sigma a}\sin\theta + 2\sin^2\theta\sin\chi\frac{u}{\sigma a}\right). \qquad (12.2)$$

Objection to this formula rests also upon the degree of convergence. In computing u and v we have to differentiate ϕ and ψ with regard to θ and the differentiation of the Fourier series makes the convergence relatively slow, though not so slow as in (12.1). This difficulty, by the way, is not due to the use of the Fourier expansions but is inherent. Differentiation with regard to χ is not prohibitive, because the convergence with regard to χ is extremely rapid.

Thirdly, we can use the first equation of Part I (2.2), giving

$$\frac{\partial}{\partial\theta}\left(\frac{\zeta - \bar{\zeta}}{h}\right) = \tfrac{1}{4}\beta\left(-i\frac{u}{\sigma a} + 2\sin\theta\sin\chi\frac{v}{\sigma a}\right). \qquad (12.3)$$

This formula is the best of all, as integration with regard to θ counteracts the prior differentiation. It is this formula which has been used for the values of ζ, as given in Tables XXV (*a*) to (*d*).

It will be noted that in the Fourier expansions so resulting we have odd and even values of s together. Special reference needs to be made to $s = 0$:—

(*a*) The term associated with cos $s\theta$; this is introduced as an integration constant to make $\zeta = 0$ at $\theta = 0$.

(*b*) The term associated with sin $s\theta$; this has a coefficient proportional to θ, and it arises quite naturally in the process of integration.

The term with coefficient θ is of very great importance ; in consequence of it we see that $\partial\zeta/\partial\theta$ is not zero at $\theta = \pi/2$, the bounding meridian. This non-zero gradient is not indicated by (12.1) because all the terms are symmetrical with regard to $\theta = \pi/2$. In other words the first formula fails near the boundary.

A consideration of the properties of the Associated Legendre Functions, whether in general terms or in connexion with the Fourier expansions, shows that, when n is not comparable with r, the function oscillates something like sin $r\theta$. This suggests that the difference between the true value of ζ and its expansion in these functions will be oscillatory and that in the main the oscillations will be more frequent as the number of terms is increased. Hence, if (12.1) were used, it would be necessary to use a graphical process to eliminate these oscillations.

The following table gives a comparison between

(*a*) the contributions to ζ on the central meridian for $\beta = 20$, $n = 1$ as obtained from the adopted method, using (12.3), and

(*b*) the corresponding values obtained by the use of (12.1).

θ	(*a*)	(*b*)	Difference	θ	(*a*)	(*b*)	Difference
0° . .	0·000	0·000	0·000	50° . .	−0·894	−0·929	0·035
10 . .	−0·412	−0·350	−0·062	60 . .	−0·916	−0·912	−0·004
20 . .	−0·724	−0·758	0·034	70 . .	−0·987	−0·949	−0·038
30 . .	−0·898	−0·909	0·011	80 . .	−1·320	−1·408	0·088
40 . .	−0·920	−0·874	−0·046	90 . .	−2·004	−1·757	−0·247

It is clear, on graphing the difference, that the discrepancies are oscillatory until near the pole, and that a graph of (*b*) smoothing out the oscillations, would give a fairly accurate representation of (*a*). The real difficulty, of course, is to decide what the discrepancy would be at the pole.

13—Amplitudes, Phase-lags, and Cotidal Charts

From the values of ζ in the complex form we deduce the form

$$\zeta = \zeta_1 \cos \sigma t + \zeta_2 \sin \sigma t, \qquad (13.1)$$

and thence

$$\zeta = \mathrm{R} \cos (\sigma t - \gamma), \qquad (13.2)$$

where γ is the lag of phase of the diurnal tide behind the phase of the diurnal equilibrium tide on the central meridian.

The values of ζ_1, ζ_2, R, γ, are given in Tables XXVI (*a*) to (*d*), and the cotidal and corange lines are drawn in figs. II to V. The case of infinite depth ($\beta = 0$), for which ζ takes the equilibrium form, is also illustrated in fig. 1.

The diagrams for the oceans are drawn without respect to systems of projections, as though θ, χ were two-dimensional polar coordinates.

14—Discussion of Results

From §10 we see that there is direct comparability between the results for $10 \cdot 948 < \beta < 30 \cdot 63$, in that $p_1{}^1$ is positive throughout and we assumed a positive value for the illustration of the resonant cases. To compare with results for $\beta = 0$ and 40 the values of γ in the resonant cases must be changed by 180°.

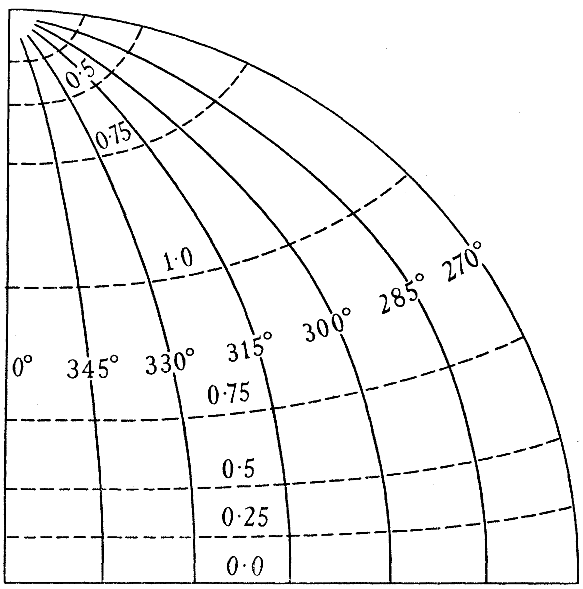

Fig. 1—Diurnal tide (K_1); cotidal and corange lines for $\beta = 0$.

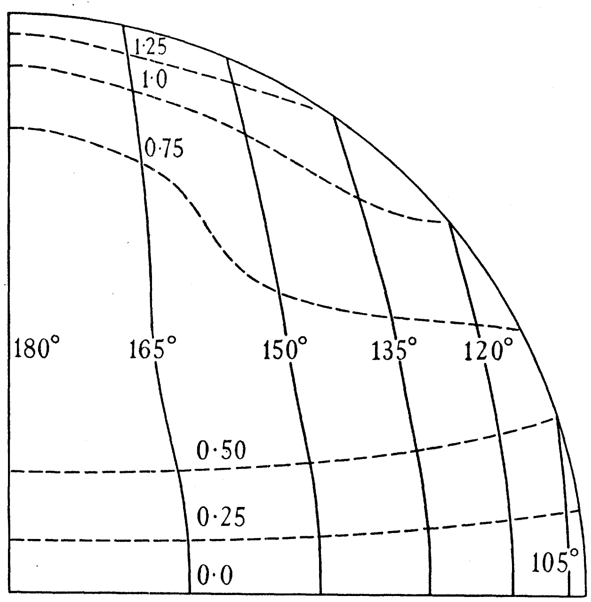

Fig. 2—Diurnal tide (K_1); cotidal and corange lines for $\beta = 10 \cdot 948$.

The first result of decreasing depth (figs. 1 and 2) is to affect the tide at the pole to a very marked degree. A stable condition is quickly reached in which the amplitude at the pole (instead of being zero as in the case of infinite depth) is the maximum for the whole ocean. The cotidal lines consequently no longer converge on the pole, and in place of the bounding meridian being a cotidal line there is a steady change in phase from equator to pole.

After the first case of resonance the changes are comparatively small, but proceed towards the development of an amphidromic point which is shown in the resonant case for $\beta = 30 \cdot 63$. The development of a point of zero range is apparent even for $\beta = 10 \cdot 948$ (fig. 2) and is still more apparent for $\beta = 20$. We may note that the phase-change along the equator, from one side of the ocean to the other, greatly decreases from $\beta = 0$ to $\beta = 30 \cdot 68$.

The genesis of the amphidromic point is of interest. In certain theoretical cases (such as the tides in non-rotating oceans), such a point develops at the pole and

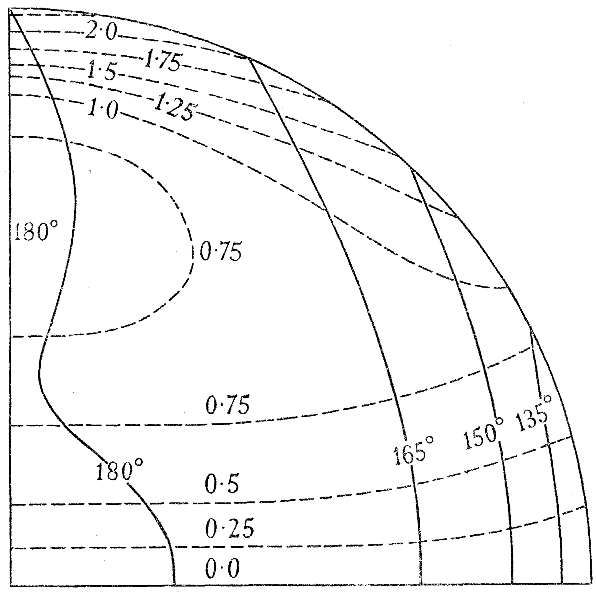

FIG. 3—Diurnal tide (K_1): cotidal and corange lines for $\beta = 20$.

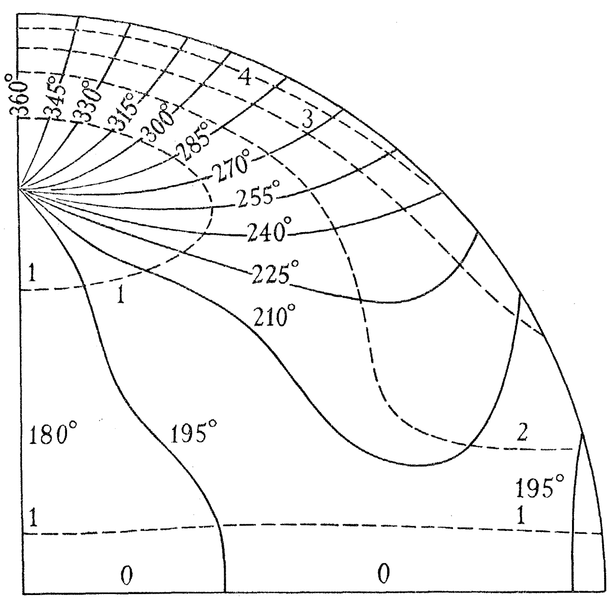

FIG. 4—Diurnal tide (K_1): cotidal and corange lines for $\beta = 30 \cdot 63$.

travels towards the equator as the depth varies, but there is no indication here that such a point is ever found near the pole after $\beta = 0$. The stability of the association between the pole and the maximum amplitudes is sufficient proof.

Without further calculation for some value of β between 30·63 and 40 it would be difficult to say whether the amphidromic point ever approaches the equator. To compare figs. 4 and 5 it is needful to add 180° to the angles in the former figure, and then we see that there is a possibility of some such phenomenon. It should be said, however, that the complexity of the cotidal lines near the equator in the case of $\beta = 40$ is somewhat misleading, for the amplitude of tide is really very small below latitude 30°, and its variations are of negligible importance. Moreover, the errors of computation may be of comparable magnitude with such a small tide and for such a large value of β. That the changes with β beyond the second resonant case tend to become complicated is apparent from §10, for the values of p_1^1 change rapidly in sign and magnitude while ip_2^2 remains steady in sign and slowly diminishes in

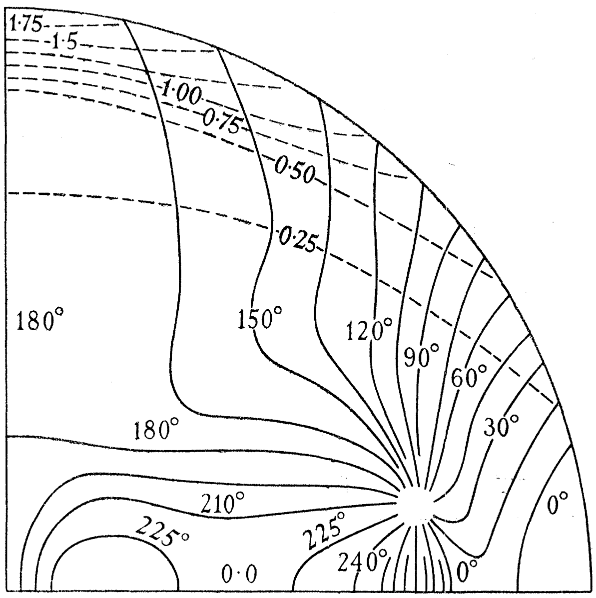

FIG. 5—Diurnal tide (K_1) : cotidal and corange lines for $\beta = 40$.

magnitude. We can safely say, however, that the principal characteristic of the tide for $\beta = 40$ remains as the concentration of the tide in polar regions.

15—CONTOUR LINES

As a preliminary discussion (it is hoped) of the physical processes, the results for $\beta = 20$ have been expressed in an alternative form. From the values of ζ_1 and ζ_2 it is easy to compute ζ for any given value of t, and this has been done for intervals of 30° in σt, corresponding approximately to time intervals of two hours of solar time. A graphical representation of the contour lines at intervals of 0·5H in ζ has been given in fig. 6.

We note that the maximum elevation at any moment is always on the bounding meridian and varies considerably. At $\sigma t = 0$ the maximum elevation is at the South

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Pole, and low water occurs at the North Pole. The maximum depression swings round the boundary towards the west and tends to fill up, so that at $\sigma t = 90°$ (the

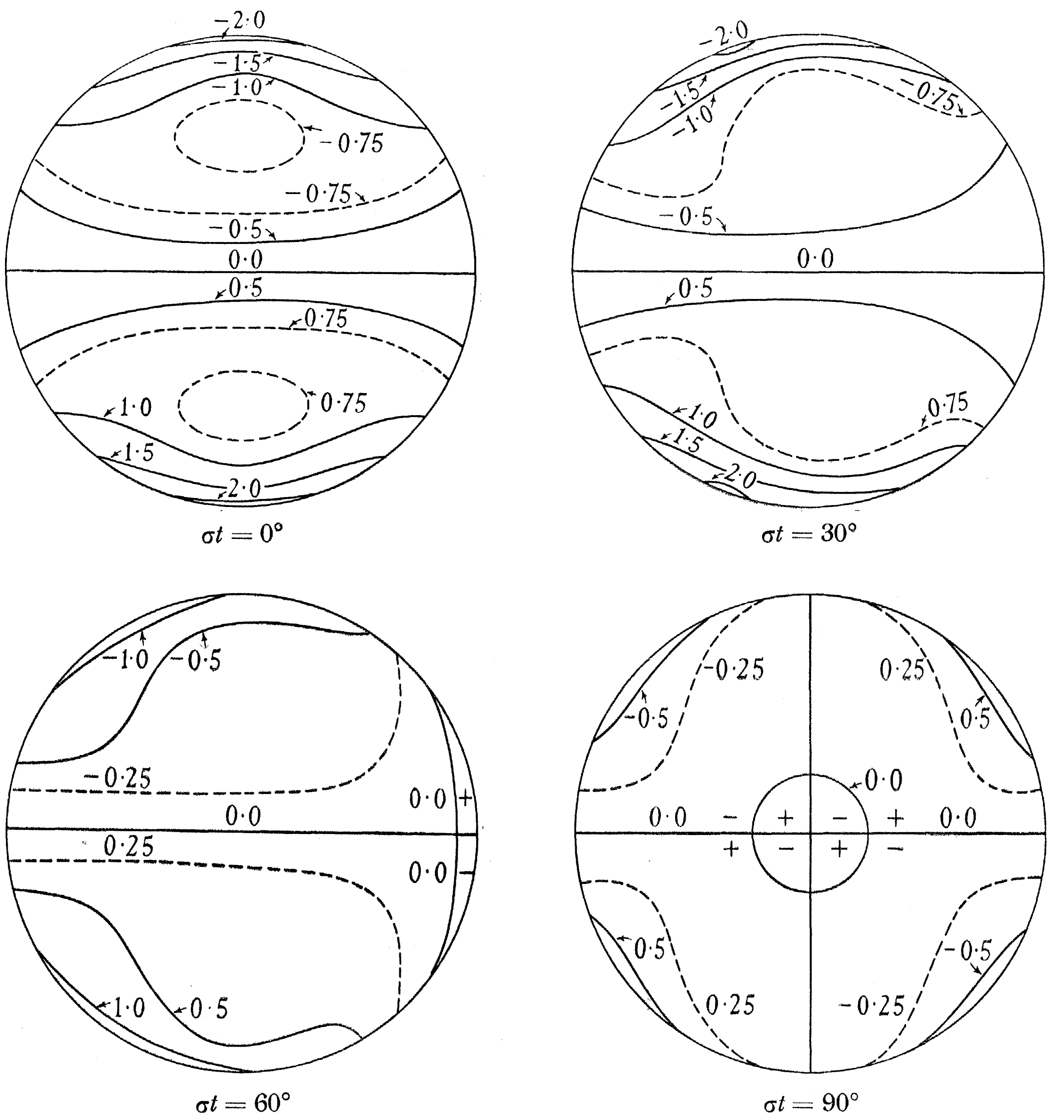

FIG. 6—Diurnal tide (K_1) : contour lines for $\beta = 20$ at intervals of two hours.

quarter period) the maximum depression or elevation is only $0{\cdot}5H$ as against $2H$ at $\sigma t = 0°$. From this point the maximum depression steadily travels south until $\sigma t = 180°$.

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Table I

$$p_{-r}^{\ n} = \Pi_{-r}^{\ n} + 2 \sum_{s,m} \beta_{-r,-s}^{\ n,m}\, i\, \Pi_{-s}^{\ m} - 4 \sum_{s,m} \beta_{-r,-s}^{\ n,m}\, p_{-s}^{\ m}$$

(Coefficients are given to 4 decimal places.)

			n = 1						n = 3					n = 5			
	m	s	r = 2	4	6	8	10	12	4	6	8	10	12	6	8	10	12
Coefficients of $i\Pi_{-s}^{\ m}$	0	1	4841	−1083	488	−279	181	−127									
		3	3019	2278	−609	304	−186	127									
		5	−748	1806	1508	−431	226	−144									
		7	358	−500	1289	1128	−334	181									
		9	−213	257	−375	1002	902	−273									
		11	142	−161	200	−300	819	751									
	2	3	−1949	−1471	393	−197	120	−82	1297	−373	191	−118	81				
		5	511	−1234	−1030	294	−155	98	1088	977	−286	152	−97				
		7	−249	347	−895	−783	232	−125	−306	849	761	−228	124				
		9	149	−180	262	−700	−630	191	158	−249	680	619	−189				
		11	−100	113	−140	211	−575	−527	−99	133	−205	564	520				
	4	5							−942	−846	248	−131	84	725	−230	126	−82
		7							287	−796	−714	214	−116	682	664	−205	113
		9							−153	240	−655	−596	182	−205	610	571	−177
		11							97	−130	200	−550	−508	111	−186	540	493
	6	7												−579	−565	174	−96
		9												190	−565	−529	164
		11												−106	177	−503	−471
Coefficients of $p_{-s}^{\ m}$	1	2	3746	214	−105	64	−46	43	−186	96	−59	40	−32				
		4	214	1382	36	−24	20	−25	−340	−30	19	−14	14				
		6	−105	36	683	15	−15	23	−28	−200	−10	8	−10				
		8	64	−24	15	400	14	−27	17	−10	−124	−7	10				
		10	−46	20	−15	14	254	36	−13	8	−7	−80	−12				
		12	43	−25	23	−27	36	107	12	−10	10	−13	−34				
	3	4	−186	−340	−28	17	−13	12	400	79	−49	35	−31	−44	30	−21	18
		6	96	−30	−200	−10	8	−10	79	332	27	−21	23	−122	−16	12	−13
		8	−59	19	−10	−124	−7	10	−49	27	224	16	−21	−15	−97	−9	11
		10	40	−14	8	−7	−80	−13	35	−21	16	151	26	11	−9	−70	−13
		12	−32	14	−10	10	−12	−34	−31	23	−21	26	65	−11	11	−14	−30
	5	6							−44	−122	−15	11	−11	143	34	−25	25
		8							30	−16	−97	−9	11	34	157	19	−23
		10							−21	12	−9	−70	−14	−25	19	124	27
		12							18	−13	11	−13	−30	25	−23	27	55

TABLE II

$$\Pi_{-r}^{\;n} = 2 \sum_{s,\,m} \beta_{-r,\,s}^{\;n,\,m}\, ip_s^{\,m}.$$

(Coefficients of ip_s^m to 4 places of decimals.)

		n = 1						n = 3					n = 5			
m	s	r = 2	4	6	8	10	12	4	6	8	10	12	6	8	10	12
2	2	−2083	2096	−714	384	−243	168	5546	−1593	0816	−505	346				
	4	−5929	−1342	2988	−1068	604	−398	3815	4724	−1451	785	−506				
	6	2523	−5345	−956	3348	−1228	714	−3535	2811	4551	−1479	829				
	8	−1707	2274	−5061	−739	3546	−1322	1814	−3841	2192	4474	−1510				
	10	1312	−1547	2140	−4894	−601	3672	−1294	1869	−3960	1790	4429				
	12	−1073	1198	−1452	2056	−4783	−506	1022	−1317	1860	−4023	1511				
4	4							−4872	2500	−1097	658	−444	6422	−2043	1116	−723
	6							−5809	−3030	2849	−1170	705	4381	5304	−1696	948
	8							2203	−5539	−2278	3128	−1261	−3019	3561	4996	−1644
	10							−1442	2174	−5303	−1833	3318	1578	−3418	2942	4838
	12							1096	−1437	2107	−5133	−1536	−1125	1703	−3607	2496
6	6												−5874	2693	−1323	839
	8												−5758	−3940	2780	−1247
	10												2050	−5639	−3108	2993
	12												−1306	2105	−5451	−2587

TABLE III

$$i\Pi_{-r}^{\ n} = -2\sum_{s,m} \beta_{-r,s}^{\ n,m}\, p_s^{\ m}.$$

(Coefficients of p_s^m to 4 places of decimals.)

		n = 0						n = 2					n = 4				n = 6		
m	s	r = 1	3	5	7	9	11	3	5	7	9	11	5	7	9	11	7	9	11
1	1	−3750	−2339	579	−277	165	−110	−4529	1187	−578	346	−231							
	3	8592	−223	−4425	1430	−773	495	−3315	−4232	1262	−667	422							
	5	−4039	7277	−55	−4978	1719	−973	3974	−2236	−4225	1364	−756							
	7	2778	−3293	6883	−22	−5242	1872	−2009	4149	−1667	−4229	1431							
	9	−2138	2279	−3040	6679	−10	−5397	1429	−1985	4196	−1325	−4232							
	11	1744	−1775	2093	−2903	6553	−6	−1125	1396	−1940	4215	−1099							
3	3							3907	−2342	934	−536	352	−6084	1855	−986	0626			
	5							5854	2333	−2904	1122	−658	−4147	−5059	1590	−875			
	7							−2329	5461	1701	−3223	1244	3236	−3235	−4800	1570			
	9							1549	−2220	5200	1341	−3420	−1678	3605	−2607	−4673			
	11							−1185	1487	−2124	5026	1108	1198	−1777	3767	−2174			
5	5												5469	−2611	1223	−757	−6656	2185	−1220
	7												5779	3545	−2809	1211	−4553	−5490	1781
	9												−2115	5596	2736	−3053	2853	−3818	−5155
	11												1364	−2137	5385	2242	−1500	3266	−3219

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TABLE IV

$p_{-r}^{\ n}$ IN TERMS OF $\Pi_{-s}^{\ m}$ AND $i\Pi_{-s}^{\ m}$.
(Coefficients to 4 places of decimals.)

		$n=1$						$n=3$					$n=5$			
m	s	$r=2$	4	6	8	10	12	4	6	8	10	12	6	8	10	12
Coefficients of $\Pi_{-s}^{\ m}$																
1	2	1·6022	408	-184	107	-75	69	-321	160	-98	66	-51	-2			
	4	406	1·1629	42	-27	24	-29	-418	-36	21	-16	16				
	6	-180	42	1·0739	16	-16	25	-31	-223	-10	9	-11	2			
	8	105	-26	15	1·0418	14	-27	17	-10	-132	-7	10		1		
	10	-74	22	-15	14	1·0261	36	-13	8	-7	-83	-12			1	
	12	67	-28	24	-28	37	1·0108	12	-10	10	-13	-34				
3	4	-320	-418	-28	16	-13	12	1·0436	83	-48	36	-31	-46	31	-21	18
	6	158	-34	-224	-10	8	-10	83	1·0351	27	-21	23	-128	-16	12	-13
	8	-94	22	-10	-132	-7	10	-50	27	1·0232	16	-21	-15	-101	-9	11
	10	63	-16	9	-7	-83	-13	35	-21	16	1·0154	26	11	-9	-72	-13
	12	-49	16	-11	10	-12	-34	-32	23	-21	26	1·0065	-11	11	-14	-30
5	6		2	3				-47	-128	-15	11	-11	1·0147	35	-25	25
	8		-1		1			31	-17	-101	-9	11	34	1·0160	19	-23
	10	1	1			1		-22	13	-9	-72	-14	-25	19	1·0126	27
	12	-1	-1					19	-14	11	-13	-30	25	-23	27	1·0055
Coefficients of $i\Pi_{-s}^{\ m}$																
0	1	7699	-1057	430	-233	144	-89	-112	71	-47	33	-27	-1			
	3	4949	2768	-698	341	-206	138	-190	53	-28	17	-11				
	5	-1159	2079	1640	-460	238	-150	-58	-51	15	-8	5				
	7	544	-566	1377	1181	-346	185	8	-23	-20	6	-3				
	9	-320	289	-400	1042	929	-280	-3	4	-11	-10	3				
	11	213	-183	212	-311	842	766	1	-1	2	-6	-5				
2	3	-3243	-1839	455	-220	132	-88	1473	-408	205	-125	82	-1	3	-3	3
	5	774	-1471	-1144	317	-164	102	1186	1055	-305	161	-100	-17	4	-2	
	7	-364	407	-976	-832	243	-130	-323	894	795	-237	128	-11	-10	2	-1
	9	213	-211	283	-737	-654	197	164	-258	704	636	-192	2	-7	-5	1
	11	-142	131	-150	221	-594	-541	-103	138	-210	577	529	-1	2	-4	-3
4	5	13	44	21	-4	1		-997	-891	257	-135	84	748	-235	127	-80
	7	-13	-12	18	10	-2	1	298	-834	-741	220	-117	705	685	-209	113
	9	10	5	-5	9	6	-1	-155	250	-678	-611	185	-211	627	586	-178
	11	-8	-3	3	-2	5	3	96	-134	206	-563	-516	114	-190	552	502
6	7								9	6	-1		-591	-575	176	-96
	9							-1	-2	6	4	-1	193	-574	-536	165
	11							1	1	-2	4	2	-107	181	-511	-474

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Table V

$$ip_{-r}^{n} = i\Pi_{-r}^{n} - 2\sum_{s,m} \beta_{-r,-s}^{n,m}\, p_{-s}^{m}.$$

(Coefficients of p_{-s}^{m} to 4 places of decimals.)

m	s	$n=0$						$n=2$					$n=4$				$n=6$		
		$r=1$	3	5	7	9	11	3	5	7	9	11	5	7	9	11	7	9	11
1	2	4841	3019	−748	358	−213	142	−1949	511	−249	149	−100							
	4	−1083	2278	1806	−500	257	−161	−1471	−1234	347	−180	113							
	6	488	−609	1508	1289	−375	200	393	−1030	−895	262	−140							
	8	−279	304	−431	1128	1002	−300	−197	294	−783	−700	211							
	10	181	−186	226	−334	902	819	120	−155	232	−630	−575							
	12	−127	127	−144	181	−273	751	−82	98	−125	191	−527							
3	4							1297	1088	−306	158	−99	−942	287	−153	97			
	6							−373	977	849	−249	133	−846	−796	240	−130			
	8							191	−286	761	680	−205	248	−714	−655	200			
	10							−118	152	−228	619	564	−131	214	−596	−550			
	12							81	−97	124	−189	520	84	−116	182	−508			
5	6												725	682	−205	111	−579	190	−106
	8												−230	664	610	−186	−565	−565	177
	10												126	−205	571	540	174	−529	−503
	12												−82	113	−177	493	−96	164	−471

TABLE VI

$$p_r^n = \Pi_r^n + x \cdot \frac{20}{\lambda_r}\left\{\tfrac{1}{2} p_r^n + \sum_{s,m} (-\beta_{r,s}^{n,m}\, ip_s^m) + \sum_{s,m} (-\beta_{r,-s}^{n,m}\, ip_{-s}^m)\right\}.$$

(Coefficients to 4 places of decimals; except $n = 1$, $r = 1$, given to 5 decimals.)

			n = 1						n = 3					n = 5			
	m	s	r = 1	3	5	7	9	11	3	5	7	9	11	5	7	9	11
Coefficients of $x \frac{20}{\lambda_r} ip_s^m$	2	2	-24206	2465	-1255	880	-682	559	2387	-1252	879	-682	558				
		4	11482	110	1044	-501	357	-282	-2123	178	-336	291	-248				
		6	-7714	196	246	655	-301	214	1217	-997	-80	-136	142				
		8	5835	-186	47	242	473	-210	-892	539	-649	-137	-61				
		10	-4699	161	-73	0	220	369	709	-401	335	-477	-148				
		12	3936	-140	73	-32	-18	197	-590	325	-250	237	-375				
	4	4							-2247	1885	-1164	880	-713	2107	-1219	903	-725
		6							1250	-328	942	-551	422	-1710	412	-445	379
		8							-905	407	-28	632	-353	1112	-890	101	-229
		10							716	-347	192	61	472	-861	551	-612	-5
		12							-594	297	-187	103	93	708	-432	362	-464
	6	6												-1984	1575	-1063	838
		8												1181	-461	844	-545
		10												-890	466	-153	591
		12												724	-397	254	-37
Coefficients of $x \frac{20}{\lambda_r} ip_{-s}^m$	0	1	18750	-4296	2020	-1389	1069	-0872									
		3	11693	112	-3638	1646	-1140	887									
		5	-2897	2213	28	-3441	1520	-1047									
		7	1387	-715	2489	11	-3340	1451									
		9	-824	387	-859	2621	5	-3277									
		11	549	-248	486	-936	2698	3									
	2	3	22643	1657	-1987	1005	-714	562	-1954	-2927	1165	-775	592				
		5	-5937	2116	1118	-2074	993	-698	1171	-1167	-2731	1110	-744				
		7	2888	-631	2112	833	-2098	970	-467	1452	-851	-2599	1062				
		9	-1729	333	-682	2114	663	-2108	268	-561	1611	-671	-2513				
		11	1156	-211	378	-715	2116	549	-176	329	-622	1710	-554				
	4	5							3042	2074	-1618	839	-599	-2735	-2889	1058	-682
		7							-928	2530	1618	-1802	888	1305	-1772	-2798	1068
		9							493	-795	2400	1303	-1884	-612	1405	-1368	-2692
		11							-313	438	-785	2336	1087	378	-606	1527	-1121
	6	7												3328	2276	-1427	750
		9												-1092	2745	1909	-1633
		11												610	-891	2578	1610
$20/\lambda_r$			10·00000	1·6667	6667	3571	2222	1515	1·6667	6667	3571	2222	1515	6667	3571	2222	1515

TABLE VII

$$ip_r^n = i\Pi_r^n + x \cdot \frac{20}{\lambda_r}\left\{\tfrac{1}{2}\,ip_r^n + \sum_{s,m}\beta_{r,s}^{n,m}\,p_s^m + \sum_{s,m}\beta_{r,-s}^{n,m}\,p_{-s}^m\right\}.$$

(Coefficients to 4 places of decimals.)

			n = 2						n = 4					n = 6			
	m	s	r = 2	4	6	8	10	12	4	6	8	10	12	6	8	10	12
Coefficients of $x\frac{20}{\lambda_r}p_s^m$	1	1	−2421	1148	−771	583	−470	394									
		3	2465	110	196	−186	161	−140									
		5	−1255	1044	246	47	−73	73									
		7	880	−501	655	242	0	−32									
		9	−682	357	−301	473	220	−18									
		11	559	−282	214	−210	369	197									
	3	3	2387	−2123	1217	−892	709	−590	−2247	1250	−905	716	−594				
		5	−1252	178	−997	539	−401	325	1885	−328	407	−347	297				
		7	879	−336	−80	−649	335	−250	−1164	942	−28	192	−187				
		9	−682	291	−136	−137	−477	237	880	−551	632	61	103				
		11	558	−248	142	−61	−148	−375	−713	422	−353	472	93				
	5	5							2107	−1710	1112	−861	708	−1984	1181	−890	724
		7							−1219	412	−890	551	−432	1575	−461	466	−397
		9							903	−445	101	−612	362	−1063	844	−153	254
		11							−725	379	−229	−5	−464	838	−545	591	−37
Coefficients of $x\frac{20}{\lambda_r}p_{-s}^m$	1	2	1042	2965	−1261	854	−656	536									
		4	−1048	671	2672	−1137	774	−599									
		6	357	−1494	478	2530	−1070	726									
		8	−192	534	−1674	369	2447	−1028									
		10	122	−302	614	−1773	300	2392									
		12	−84	199	−357	661	−1836	253									
	3	4	−2773	−1907	1768	−907	647	−511	2436	2904	−1102	721	−548				
		6	797	−2362	−1406	1921	−934	658	−1250	1515	2770	−1087	718				
		8	−408	726	−2275	−1096	1980	−930	549	−1424	1139	2652	−1054				
		10	253	−392	739	−2237	−895	2012	−329	585	−1564	917	2567				
		12	−173	253	−414	755	−2215	−756	222	−352	630	−1659	768				
	5	6							−3211	−2190	1509	−789	562	2937	2879	−1025	653
		8							1021	−2652	−1780	1709	−851	−1346	1970	2820	−1053
		10							−558	848	−2498	−1471	1804	662	−1390	1554	2726
		12							361	−474	822	−2419	−1248	−419	623	−1497	1293
$20/\lambda_r$			3·3333	1·0000	4762	2778	1818	1282	1·0000	4762	2778	1818	1282	4762	2778	1818	1282

TABLE VIII

COEFFICIENTS OF $\frac{H}{h}$ IN EXPANSIONS FOR Π_{-r}^{n}, $i\Pi_{-r}^{n}$, ip_{r}^{n}, ip_{-r}^{n}, p_{r}^{n} AND p_{-r}^{n}.

(4 places of decimals.)

A : Π_{-r}^{n} and $i\Pi_{-r}^{n}$

Part	n	r	1	x	x^2	x^3
(a)	1	2		5	− 1	
		4		− 8	2	
		6		3	− 1	
		8		3	0	
		10		− 3	1	
		12		3	− 1	
	3	4		− 6	1	
		6		8	− 3	
		8		2	0	
		10		− 4	2	
		12		4	− 1	
	5	6		0	0	
		8		− 2	2	
		10		− 1	1	
		12		1	− 1	
(b)	0	1	−167	41	−16	5
		3	272	−64	25	− 8
		5	− 92	35	−10	3
		7	−111	28	−12	4
		9	87	−38	13	− 4
		11	− 75	20	− 7	2
	2	3	152	−61	21	− 7
		5	−124	44	−16	5
		7	− 86	28	−11	4
		9	66	−42	14	− 5
		11	− 56	22	− 8	3
	4	5		22	−10	3
		7		5	0	0
		9		−17	6	− 2
		11		9	− 4	1
	6	7			4	− 1
		9			1	0
		11			− 1	0

B : ip_{r}^{n} and ip_{-r}^{n}

Part	n	r	1	x	x^2	x^3
(a)	2	2	...	...	...	...
		4	...	...	...	...
		6		12	− 2	
		8		− 6	2	
		10		3	− 1	
		12		− 1	0	
	4	4	...	...	...	...
		6		− 1	1	
		8		− 2	2	
		10		1	− 1	
		12		0	0	
	6	6				
		8				
		10				
		12				
(b)	0	1	−199	57	−21	7
		3	276	−61	24	− 8
		5	− 81	30	− 8	3
		7	−119	31	−13	4
		9	91	−40	14	− 5
		11	− 78	21	− 7	2
	2	3	152	−65	22	− 7
		5	−134	47	−17	6
		7	− 83	28	−11	4
		9	64	−42	14	− 5
		11	− 55	22	− 8	3
	4	5	2	22	−10	3
		7	2	3	1	0
		9	− 1	−16	6	− 2
		11	1	9	− 4	1
	6	7			4	− 1
		9			1	0
		11			− 1	0

C : p_{r}^{n} and p_{-r}^{n}

Part	n	r	1	x	x^2	x^3
(a)	1	1	...	...	...	...
		3	...	...	...	...
		5	306	−65	28	− 9
		7	−100	48	−14	5
		9	50	− 8	3	− 1
		11	− 28	− 3	1	− 1
	3	3	...	...	...	...
		5		−30	8	− 3
		7		27	− 9	4
		9		− 4	2	− 1
		11		0	1	0
	5	5			− 3	1
		7			− 3	1
		9			1	− 1
		11			0	0
(b)	1	2	− 47	28	− 9	3
		4	69	−20	8	− 3
		6	− 29	9	− 3	1
		8	9	1	1	0
		10	− 4	− 3	1	0
		12	− 2	3	− 1	0
	3	4	9	−13	3	− 1
		6	− 28	16	− 6	2
		8	6	0	0	0
		10	− 1	− 4	2	− 1
		12	− 3	5	− 1	0
	5	6		2	− 1	0
		8		− 3	2	− 1
		10		− 1	1	0
		12		2	− 1	0

Table IX

Coefficients of p_1^1 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.
(4 places of decimals.)

B : ip_r^n and ip_{-r}^n

(a)

n	r	1	x	x^2	x^3	x^4	x^5
2	2	...	...	...	...	...	...
	4	...	...	...	...	...	...
	6		−197	0	−11	2	−1
	8		95	−9	5	−2	1
	10		−52	1	−2	0	0
	12		32	3	1	0	0
4	4	...	...	...	...	...	...
	6		−31	3	−3	1	−1
	8		20	−14	4	−2	0
	10		−12	2	−1	0	0
	12		9	2	0	0	0
6	6			−9	2	−1	0
	8			−6	2	−1	0
	10			2	−1	0	0
	12			−1	0	0	0

(b)

n	r	1	x	x^2	x^3	x^4	x^5
0	1	−5053	−484	157	−57	21	−8
	3	−2969	405	−203	62	−25	9
	5	821	−144	71	−27	9	−3
	7	−405	−190	111	−35	14	−5
	9	245	164	−107	38	−13	6
	11	−165	−155	56	−19	7	−3
2	3	−4187	820	−158	73	−26	9
	5	1002	−257	134	−44	17	−6
	7	−471	−282	95	−36	13	−5
	9	277	246	−111	45	−16	6
	11	−182	−226	53	−22	8	−3
4	5	17	−406	107	−42	17	−6
	7	−17	−193	0	−6	1	−1
	9	13	159	−58	24	−8	3
	11	−10	−133	24	−13	5	−2
6	7		8	−68	16	−7	3
	9		−4	−21	0	−1	0
	11		1	9	−4	1	−1

C : p_r^n and p_{-r}^n

(a)

n	r	1	x	x^2	x^3	x^4	x^5
1	1	1·0000	...	...	...	...	...
	3	...	...	...	...	...	...
	5		501	−230	71	−28	10
	7		−209	109	−43	15	−6
	9		109	−20	6	−2	1
	11		−71	−9	4	−1	1
3	3	...	...	...	...	...	...
	5		677	−37	40	−11	5
	7		−238	70	−29	11	−4
	9		114	−4	5	−2	1
	11		−66	−13	3	−1	0
5	5		−5	54	−13	6	−2
	7		0	64	−11	6	−2
	9		1	−7	6	−2	1
	11		−1	−3	1	0	0

(b)

n	r	1	x	x^2	x^3	x^4	x^5
1	2	−2542	−350	50	−28	9	−3
	4	516	168	−49	18	−7	3
	6	−222	−55	22	−7	3	−1
	8	124	−33	−4	−1	1	0
	10	−79	40	−3	2	−1	0
	12	53	−42	1	−2	0	0
3	4	−417	244	−26	18	−6	2
	6	205	−144	30	−14	4	−1
	8	−120	−33	1	−3	1	−1
	10	81	51	−12	6	−2	1
	12	−62	−61	7	−4	1	0
5	6	0	−71	28	−10	4	−1
	8	−1	5	−6	1	0	−1
	10	1	9	−9	4	−1	0
	12	−1	−18	6	−3	0	0

Table X

Coefficients of ip_2^2 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.
(4 places of decimals.)

B: ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
2	2	1·0000	...	...	...	...	...
	4	...	...	...	...	...	...
	6		902	−93	40	−10	3
	8		−398	74	−20	6	−2
	10		214	−12	5	−1	0
	12		−135	−14	0	−1	0
4	4	...	...	...	...	...	...
	6		601	−52	31	−4	2
	8		−247	85	−24	6	−2
	10		129	−10	7	−1	1
	12		−80	−14	1	−1	0
6	6		−2	38	5	5	1
	8		1	56	−11	3	−1
	10		0	−10	7	−1	1
	12		0	−1	−1	0	0
0	1	−1950	1184	−356	125	−46	16
	3	−497	−937	432	−135	54	−19
	5	512	140	−117	48	−18	7
	7	−267	586	−269	83	−32	12
	9	166	−396	226	−82	30	−11
	11	−116	302	−104	38	−14	5
2	3	1151	−1560	322	−151	51	−19
	5	82	527	−273	92	−36	13
	7	−50	638	−218	79	−29	11
	9	33	−475	227	−92	33	−13
	11	−25	391	−92	43	−16	6
4	5	−376	776	−240	91	−33	11
	7	222	243	44	3	1	1
	9	−145	−232	88	−46	15	−7
	11	104	210	−29	27	−7	4
6	7	−1	−116	188	−38	19	−4
	9		33	−1	3	−1	−1
	11		−6	...	8	−1	1

C: p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5	
1	1	...	...	...	...	...	...	
	3	...	...	...	...	...	...	
	5		−1191	499	−155	62	−22	
	7		375	−215	87	−31	12	
	9		−170	21	−8	4	−1	
	11		88	23	−9	3	−1	
3	3	...	...	...	...	...	...	(a)
	5		−1076	52	−75	21	−9	
	7		377	−129	60	−21	8	
	9		−183	−2	−10	3	−1	
	11		104	28	−5	3	−1	
5	5		96	−200	30	−20	3	
	7		15	−95	23	−9	4	
	9		−15	12	−16	3	−2	
	11		12	5	2	1	0	
1	2	−3448	955	−152	66	−21	7	
	4	2124	−783	177	−54	19	−6	
	6	−700	222	−68	18	−6	3	
	8	371	187	−26	9	−3	1	
	10	−232	−193	43	−11	4	−1	
	12	154	198	−21	7	−2	...	
3	4	5750	−1099	140	−64	15	−6	(b)
	6	−1625	546	−131	40	−13	4	
	8	823	340	−57	22	−5	1	
	10	−503	−347	98	−29	8	−2	
	12	339	346	−48	17	−5	1	
5	6	−7	455	−132	37	−12	3	
	8	12	157	1	4	2	0	
	10	−11	−159	71	−22	4	−2	
	12	13	153	−36	16	−2	1	

Table XI

Coefficients of p_3^1 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.
(4 places of decimals.)

B: ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
2	2	...	...	...	...	...	
	4	...	...	...	...	...	...
	6		−554	75	−29	8	−3
	8		185	−46	14	−4	1
	10		−87	5	−2	1	0
	12		47	6	−1	0	0
4	4	...	...	...	...	...	...
	6		−166	72	−14	6	−1
	8		31	−50	14	−5	2
	10		−12	1	−1	1	0
	12		5	7	−2	1	0
6	6		1	−16	8	−1	1
	8		0	−23	10	−3	1
	10		0	1	−3	1	0
	12		0	1	−1	0	0

C: p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
1	1	...	...	...	...	...	...
	3	1·0000	...	...	...	...	...
	5		2095	−480	170	−61	22
	7		112	218	−76	30	−11
	9		−70	55	2	−2	1
	11		65	−32	13	−4	1
3	3	...	...	...	...	...	...
	5		1394	−180	85	−28	10
	7		98	165	−52	22	−8
	9		−49	53	1	−2	1
	11		33	−31	11	−3	1
5	5		−25	62	−33	11	−5
	7		−9	116	−29	12	−4
	9		4	22	7	−3	1
	11		−2	−12	5	−1	0

(a)

B: ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
0	1	1·2455	−1204	325	−122	45	−16
	3	1970	1385	−390	140	−51	19
	5	−5121	255	99	−40	17	−6
	7	1750	−1204	305	−95	32	−12
	9	−963	379	−227	81	−30	10
	11	622	−162	61	−33	14	−5
2	3	−4869	1710	−383	153	−56	20
	5	−3889	−58	224	−79	32	−13
	7	1071	−1369	313	−96	32	−11
	9	−549	400	−259	93	−34	12
	11	341	−170	56	−36	15	−6
4	5	116	−609	225	−92	34	−12
	7	−30	−710	95	−24	6	−2
	9	14	186	−158	51	−19	6
	11	−8	−88	24	−20	9	−3
6	7		39	−94	38	−14	5
	9		−5	−59	7	−2	1
	11		−1	7	−6	3	−1

C: p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
1	2	7812	−719	150	−59	21	−7
	4	−699	620	−134	46	−15	6
	6	83	−134	46	−16	5	−1
	8	−39	−135	21	−7	1	−1
	10	18	110	−27	8	−2	1
	12	6	−94	12	−5	1	0
3	4	−1111	650	−142	47	−16	5
	6	−97	−303	83	−30	10	−4
	8	57	−232	57	−16	5	−2
	10	−42	171	−66	20	−7	2
	12	40	−134	30	−9	3	−1
5	6	5	−184	100	−33	11	−3
	8	−3	−94	26	−2	1	0
	10	2	52	−51	15	−6	1
	12	−1	−36	22	−7	4	0

(b)

Table XII

Coefficients of p_3^3 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.
(4 places of decimals.)

B : ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
2	2	...	...	...	...	...	...
	4	...	...	...	...	...	...
	6		734	−88	41	−10	4
	8		−282	49	−16	5	−2
	10		141	1	1	0	0
	12		−79	−18	2	−1	0
4	4	...	...	...	...	...	...
	6		730	−148	44	−12	3
	8		−288	101	−25	9	−2
	10		150	−16	3	−1	0
	12		−91	−10	4	−1	0
6	6		−60	−16	−14	−2	−2
	8		−5	106	−16	6	−1
	10		8	−30	6	−2	0
	12		−8	9	1	0	0
0	1	−669	790	−293	110	−41	15
	3	−565	−413	332	−112	48	−17
	5	111	118	−89	41	−15	7
	7	−22	238	−198	70	−28	11
	9	6	−219	169	−70	27	−10
	11	−1	202	−90	32	−13	5
2	3	4390	−1396	329	−145	50	−18
	5	−2309	531	−226	81	−33	11
	7	921	333	−175	72	−28	10
	9	−527	−425	199	−85	31	−12
	11	345	433	−104	41	−15	5
4	5	−6207	1951	−427	136	−40	14
	7	1878	−245	82	−10	2	0
	9	−991	−274	118	−45	17	−6
	11	625	364	−92	28	−10	3
6	7	2	−1243	338	−84	23	−8
	9	−7	182	−16	12	0	0
	11	8	−19	−38	5	−4	1

C : p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5	
1	1	...	...	...	...	...	...	
	3	...	...	...	...	...	...	
	5		−548	394	−132	56	−20	
	7		294	−160	75	−27	11	
	9		−151	36	−8	4	−1	
	11		94	14	−7	3	−1	
3	3	1·0000	...	...	...	...	...	(a)
	5		−1031	105	−79	22	−9	
	7		707	−155	63	−22	8	
	9		−352	42	−13	4	−1	
	11		218	18	−4	2	−1	
5	5		1353	−306	88	−20	8	
	7		457	−187	32	−15	4	
	9		−211	95	−21	7	−2	
	11		125	−24	1	−1	0	
1	2	−1511	805	−143	63	−19	7	
	4	−350	−503	131	−45	16	−6	
	6	325	81	−37	11	−4	3	
	8	−187	196	−33	13	−3	1	
	10	126	−163	35	−11	3	−1	
	12	−100	135	−10	4	−1	1	
3	4	938	−1183	214	−74	21	−7	(b)
	6	50	327	−76	28	−11	4	
	8	−33	414	−103	34	−9	3	
	10	25	−354	108	−32	10	−2	
	12	−27	319	−48	15	−5	1	
5	6	−294	691	−204	60	−16	6	
	8	196	172	−61	10	−4	1	
	10	−140	−179	113	−27	9	−2	
	12	120	186	−71	15	−5	1	

Table XIII

Coefficients of ip_4^2 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.

(4 places of decimals.)

(a)

B: ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
2	2	...	...	...	...	...	
	4	1·0000	...	...	...	...	
	6		750	7	17	−1	
	8		139	−2	4	−1	
	10		−83	36	−4	1	
	12		65	−15	2	−1	
4	4	...	...	...	...	...	
	6		1064	−101	40	−6	2
	8		146	43	−6	2	−1
	10		−78	40	−2	1	
	12		52	−20	2	−1	
6	6		−12	12	6	2	
	8		−5	84	−11	3	−1
	10		2	18	3	0	
	12		−1	−9	2	0	

C: p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
1	1	...	...	...	...	...	...
	3	...	...	...	...	...	...
	5		700	−50	50	−12	5
	7		−229	138	−27	11	−3
	9		62	−29	13	−3	1
	11		−25	−2	2	0	0
3	3	...	...	...	...	...	...
	5		−560	−57	−15	−4	1
	7		−3	56	−10	5	−1
	9		12	−29	7	−2	1
	11		−10	10	2	0	0
5	5		140	−163	14	−9	3
	7		103	−77	7	−2	1
	9		−17	−5	−4	−1	0
	11		4	8	2	0	0

(b)

B: ip_r^n and ip_{-r}^n

n	r	1	x	x^2	x^3	x^4	x^5
0	1	−4255	−265	91	−33	11	−4
	3	−3609	670	−81	50	−13	4
	5	919	−279	100	−22	9	−3
	7	6	−281	9	−18	4	−1
	9	−10	251	−87	25	−8	3
	11	14	−160	50	−18	5	−2
2	3	2666	−115	−105	17	−11	4
	5	358	−504	116	−41	11	−4
	7	138	−14	−12	−3	2	−1
	9	−85	125	−86	21	−8	3
	11	59	−88	61	−16	5	−2
4	5	−860	527	−102	21	−3	1
	7	−137	310	−3	11	0	0
	9	90	−123	−15	0	−3	1
	11	−68	61	31	−1	3	−1
6	7	3	−215	155	−17	8	−3
	9	−2	−37	10	1	0	0
	11	1	21	1	2	1	0

C: p_r^n and p_{-r}^n

n	r	1	x	x^2	x^3	x^4
1	2	−9652	225	48	−2	4
	4	−1966	−120	−23	8	−3
	6	3195	−261	22	−11	3
	8	−1146	311	−23	9	−2
	10	647	−70	3	−1	0
	12	−428	−4	15	−1	0
3	4	4266	−895	58	−32	4
	6	4760	−417	55	−15	3
	8	−1422	760	−92	28	−5
	10	759	−181	41	−9	2
	12	−481	29	18	1	0
5	6	−73	488	−115	25	−7
	8	18	663	−106	26	−5
	10	−6	−155	81	−17	3
	12	0	58	−3	7	−1

Table XIV

Coefficients of ip_4^4 in Expansions for ip_r^n, ip_{-r}^n, p_r^n and p_{-r}^n.
(4 places of decimals.)

B: ip_r^n and ip_{-r}^n n	r	1	x	x^2	x^3	x^4	x^5	C: p_r^n and p_{-r}^n n	r	1	x	x^2	x^3	x^4	x^5	
2	2	...	...	...	...	...	...	1	1	...	...	...	...	...	...	
	4	...	...	...	...	...	...		3	...	...	...	...	...	...	
	6		−435	46	−19	6	−1		5		124	−296	75	−35	12	
	8		240	−31	9	−3	1		7		−17	65	−40	15	−6	
	10		−132	1	−1	0	0		9		−2	3	−4	−1	0	
	12		88	18	0	1	0		11		6	−15	6	−2	1	
4	4	1·0000	...	...	...	...	...	3	3	...	...	...	...	...	...	(a)
	6		−785	−44	−50	−2	−4		5		1594	−45	70	−14	6	
	8		556	−34	23	−1	2		7		−490	39	−43	10	−5	
	10		−299	−2	−9	−2	−1		9		209	49	3	0	1	
	12		197	31	4	3	0		11		−108	−62	−1	−4	0	
6	6		1090	−130	−40	−17	−5	5	5		1199	351	29	32	3	
	8		351	−27	15	−1	2		7		−517	118	−40	3	−5	
	10		−176	13	−10	−1	−1		9		226	31	21	1	2	
	12		109	8	4	1	0		11		−127	−55	−8	−4	−1	
0	1	80	−370	175	−60	24	−8	1	2	212	−638	64	−38	11	−3	
	3	110	−76	−235	59	−29	11		4	193	118	−92	19	−10	3	
	5	13	61	23	−19	8	−3		6	−38	−33	27	−6	3	−1	
	7	−7	−114	167	−46	18	−6		8	3	−98	16	−6	1	0	
	9	3	54	−100	40	−16	6		10	4	104	−21	6	−2	0	
	11	−1	−22	40	−14	7	−2		12	−9	−102	7	−3	1	0	
2	3	−856	1400	−154	100	−28	12	3	4	−5095	1110	−40	60	−6	5	(b)
	5	−278	59	93	−38	17	−7		6	2466	−355	57	−14	5	−2	
	7	268	−524	178	−56	19	−6		8	−1081	−233	30	−20	4	−1	
	9	−170	360	−117	52	−19	7		10	644	315	−53	20	−5	2	
	11	120	−267	13	−21	7	−3		12	−431	−343	12	−12	1	−1	
4	5	771	−777	272	−63	28	−6	5	6	6498	−1580	119	−26	13	−2	
	7	29	−911	14	−28	−1	−2		8	−2059	202	−53	−24	−4	−3	
	9	−21	705	−77	47	−7	5		10	1118	209	−26	27	0	3	
	11	19	−556	−30	−29	0	−3		12	−716	−316	3	−22	−2	−2	
6	7	−234	−392	−274	10	−23	1									
	9	169	360	−11	17	4	2									
	11	−128	−289	−21	−19	−3	−2									

Table XV

Six Simultaneous Equations in p_1^1, ip_2^2, . . . H/h, with Coefficients as Power-series in x^t.

t	p_1^1	ip_2^2	p_3^1	p_3^3	ip_4^2	ip_4^4	H/h		
0	−1·0000	...	...	...	...	...	−0·7675		
1	2·6450	−2·6114	1·9224	0·9757	0·4906	−0·1379	0·0327	= 0	
2	0·1663	−0·3401	0·3098	−0·3058	−0·0131	0·2641	−0·0136		
3	−0·0325	0·0703	−0·0754	0·0689	−0·0260	−0·0297	0·0044		
4	0·0148	−0·0315	0·0301	−0·0296	0·0046	0·0193	−0·0013		(*a*)
5	−0·0053	0·0107	−0·0110	0·0105	−0·0025	−0·0057	0·0005		
6	0·0018	−0·0040	0·0041	−0·0037	0·0008	0·0027	−0·0002		
7	−0·0006	0·0013	−0·0014	0·0012	−0·0003	−0·0009	0·0001		
0	...	−1·0000	...	...	...	...	0·5284		
1	−0·8705	0·8686	1·2175	0·6767	−0·4567	0·5593	−0·0235	= 0	
2	−0·1133	0·2955	−0·2445	0·2643	0·0512	−0·2229	0·0097		
3	0·0233	−0·0590	0·0592	−0·0601	0·0121	0·0245	−0·0033		
4	−0·0105	0·0247	−0·0225	0·0237	−0·0014	−0·0158	0·0012		(*b*)
5	0·0036	−0·0080	0·0081	−0·0080	0·0013	0·0040	−0·0004		
6	−0·0014	0·0030	−0·0029	0·0029	−0·0004	−0·0018	0·0001		
7	0·0005	−0·0010	0·0010	−0·0010	0·0001	0·0006	...		
0	...	...	−1·0000	...	...	...	−0·1954		
1	0·3205	0·6086	−0·5605	0·0772	0·4340	−0·0426	0·0150	= 0	
2	0·0516	−0·1223	0·1753	−0·0815	−0·0026	0·0769	−0·0045		
3	−0·0127	0·0297	−0·0329	0·0262	−0·0031	−0·0183	0·0017		
4	0·0051	−0·0113	0·0123	−0·0103	0·0013	0·0068	−0·0006		(*c*)
5	−0·0019	0·0040	−0·0043	0·0037	−0·0006	−0·0023	0·0002		
6	0·0007	−0·0014	0·0015	−0·0013	0·0002	0·0008	−0·0001		
7	−0·0002	0·0005	−0·0005	0·0004	−0·0001	−0·0003	...		
0	...	...	...	−1·0000	...	...	...		
1	0·1626	0·3383	0·0774	0·2796	−0·4757	−0·3169	−0·0064	= 0	
2	−0·0509	0·1321	−0·0814	0·1924	0·0455	−0·0997	0·0038		
3	0·0116	−0·0301	0·0260	−0·0411	−0·0015	0·0185	−0·0014		
4	−0·0049	0·0119	−0·0103	0·0141	0·0008	−0·0076	0·0004		(*d*)
5	0·0019	−0·0039	0·0039	−0·0044	0·0002	0·0025	−0·0001		
6	−0·0006	0·0013	−0·0014	0·0015	−0·0001	−0·0008	...		
7	0·0002	−0·0004	0·0005	−0·0005	...	0·0003	...		
0	...	...	...	...	−1·0000	...	...		
1	0·0490	−0·1370	0·2606	−0·2854	−0·0643	0·0356	0·0040	= 0	
2	−0·0013	0·0153	−0·0015	0·0274	0·0527	−0·0285	−0·0006		
3	−0·0026	0·0036	−0·0019	−0·0009	−0·0041	−0·0024	0·0003		
4	0·0005	−0·0004	0·0007	0·0005	0·0021	−0·0008	−0·0001		(*e*)
5	−0·0002	0·0004	−0·0003	0·0002	−0·0004	−0·0002	...		
6	0·0001	−0·0002	0·0001	−0·0001	0·0000	0·0001	...		
7	...	0·0001	...	...	...	...	...		
0	...	...	...	...	...	−1·0000	...		
1	−0·0138	0·1678	−0·0255	−0·1901	0·0356	0·0976	0·0006	= 0	
2	0·0264	−0·0668	0·0461	−0·0597	−0·0284	0·1518	−0·0015		
3	−0·0030	0·0073	−0·0109	0·0111	−0·0024	0·0007	0·0004		
4	0·0019	−0·0047	0·0041	−0·0045	−0·0008	0·0050	−0·0002		(*f*)
5	−0·0006	0·0012	−0·0015	0·0015	−0·0001	−0·0003	0·0001		
6	0·0002	−0·0005	0·0005	−0·0005	0·0000	0·0005	...		
7	−0·0001	0·0002	−0·0002	0·0002	...	−0·0002	...		

Terms with $t = 7$ were obtained from terms with $t = 6$ by dividing by -3 and additional terms were similarly obtained.

Table XVI

Equations Resulting from XV (e) and (f) after Eliminating either $ip_4{}^2$ or $ip_4{}^4$.

t	$p_1{}^1$	$ip_2{}^2$	$p_3{}^1$	$p_3{}^3$	$ip_4{}^2$	$ip_4{}^4$	H/h	
0	...	...	...	...		−1·0000	...	
1	−0·0138	0·1678	−0·0255	−0·1901		0·0333	0·0006	= 0
2	0·0273	−0·0609	0·0537	−0·0821		0·2120	−0·0013	
3	−0·0020	−0·0014	−0·0140	0·0264		−0·0008	0·0001	
4	0·0001	0·0000	0·0002	−0·0015		0·0002	−0·0001	
5	−0·0001	−0·0002	−0·0005	0·0010		0·0001	−0·0001	
6	−0·0002	0·0001	0·0000	−0·0001		0·0000	0·0000	
0	...	...	...	...	−1·0000		...	
1	0·0490	−0·1370	0·2606	−0·2854	0·0333		0·0040	= 0
2	−0·0066	0·0346	−0·0278	0·0485	0·2120		−0·0010	
3	−0·0086	0·0157	−0·0389	0·0430	−0·0008		−0·0003	
4	0·0001	−0·0012	−0·0007	−0·0008	0·0002		0·0000	
5	...	0·0002	−0·0010	0·0015	0·0001		−0·0001	
6	...		−0·0001	−0·0002				

Table XVII

Final Equations for $p_3{}^1$ and $p_3{}^3$ in terms of H/h, $p_1{}^1$ and $ip_2{}^2$.

t	$p_1{}^1$	$ip_2{}^2$	$p_3{}^1$	$p_3{}^3$	H/h	
0	...	...		−1·0000	...	
1	0·1626	0·3383		−0·2143	−0·0215	= 0
2	0·0352	0·3583		1·2823	0·0393	
3	−0·2066	−0·3701		0·0814	−0·0007	
4	0·0119	−0·1169		−0·4183	−0·0119	
5	0·0517	0·0873		0·0000	0·0010	
6	−0·0043	0·0095		0·0429	0·0009	
7	−0·0030	−0·0050		−0·0023	−0·0001	
8	0·0001	0·0000		−0·0001	0·0000	
0	...	...	−1·0000		−0·1954	
1	0·3205	0·6086	−0·2143		0·0826	= 0
2	−0·0249	−0·3735	1·2823		0·1464	
3	−0·3350	−0·4162	0·0814		−0·0443	
4	0·0324	0·1890	−0·4183		−0·0293	
5	0·0818	0·0754	0·0000		0·0061	
6	−0·0068	−0·0238	0·0429		0·0013	
7	−0·0045	−0·0025	−0·0023		−0·0001	
8	0·0000	0·0002	−0·0001		0·0001	

Table XVIII

Final Equations for ip_2^4 and ip_4^4 in terms of p_1^1, ip_2^2 and H/h.

t	p_1^1	ip_2^2	ip_4^2	ip_4^4	H/h	
0	...	...		−1·0000	...	
1	−0·0138	0·1678		−0·1810	0·0056	= 0
2	−0·0147	−0·1048		1·5014	−0·0097	
3	0·0194	−0·2833		0·0833	−0·0032	
4	0·0092	0·0960		−0·6928	0·0057	
5	−0·0085	0·1165		−0·0022	−0·0001	
6	−0·0007	−0·0286		0·1314	−0·0009	
7	0·0011	−0·0166		−0·0041	0·0002	
8	−0·0001	0·0031		−0·0089	0·0001	
9	−0·0002	0·0004		0·0005		
10	0·0000	0·0001		−0·0001		

t	p_1^1	ip_2^2	ip_4^2	ip_4^4	H/h	
0	...	...	−1·0000		...	
1	0·0490	−0·1370	−0·1810		−0·0469	= 0
2	0·0410	0·0673	1·5014		0·0330	
3	−0·0904	−0·0013	0·0833		0·0256	
4	−0·0287	−0·0153	−0·6928		−0·0166	
5	0·0386	0·0252	−0·0022		−0·0049	
6	0·0081	−0·0013	0·1314		0·0026	
7	−0·0064	−0·0055	−0·0041		0·0004	
8	−0·0009	0·0002	−0·0089		−0·0001	
9	0·0004	0·0003	0·0005		−0·0001	
10	0·0000	0·0001	−0·0001		0·0000	

Table XIX

Equations for p_1^1 and ip_2^2 in terms of H/h.

t	p_1^1	ip_2^2	H/h		p_1^1	ip_2^2	H/h	
0	−2·0000	...	−1·5350		...	−2·0000	1·0568	
1	4·4994	−5·2228	−1·2926	= 0	−1·7410	0·9466	−0·1051	= 0
2	7·5150	0·0747	2·6482		0·0256	7·0201	−1·6126	
3	−8·1182	9·1098	1·5472		3·0367	−2·6618	0·3256	
4	−5·4337	−1·4492	−0·4766		−0·4840	−4·6925	−0·0482	
5	1·7402	−0·9809	0·4420		−0·3257	1·7676	−0·0865	
6	−3·8514	1·1052	−1·9492		0·3682	−3·9673	1·3040	
7	5·8544	−7·0801	−1·3747		−2·3611	1·5923	−0·2300	
8	6·6381	0·8926	1·8099		0·2989	6·0120	−0·9434	
9	−5·6241	5·7636	0·7859		1·9203	−2·3473	0·2004	
10	−3·4528	−1·1551	−0·7271		−0·3855	−2·9314	0·3134	
11	2·3129	−2·0497	−0·2089		−0·6823	1·1292	−0·0716	
12	0·8847	0·4749	0·1554		0·1578	0·7099	−0·0556	
13	−0·5063	0·3828	0·0273		0·1274	−0·2715	0·0132	
14	−0·1153	−0·0959	−0·0174		−0·0319	−0·0870	0·0050	
15	0·0578	−0·0362	−0·0012		−0·0120	0·0334	−0·0012	
16	0·0067	0·0098	0·0008		0·0035	0·0043	−0·0002	
17	−0·0027	0·0012	−0·0001		0·0004	−0·0016	0·0000	
18	−0·0001	−0·0004	0·0000		−0·0002	0·0000	0·0000	
19	0·0000	0·0000	0·0000		−0·0001	0·0000	0·0000	

TABLE XX (*a*)

VALUES OF COORDINATES. $\beta = 10{\cdot}948$.

RESONANT CASE. RELATIVE VALUES ONLY.

n	r	$p_r{}^n\pi_r{}^n$	n	r	$ip_r{}^n\pi_r{}^n$	n	r	$p_{-r}{}^n\pi_r{}^n$	n	r	$ip_{-r}{}^n\pi_r{}^n$
1	1	0·5642	2	2	−0·1046	1	2	−0·0514	0	1	−0·1713
	3	0·0089		4	0·0047		4	−0·0025		3	−0·0431
	5	0·0035		6	−0·0016		6	0·0007		5	0·0045
	7	−0·0008		8	0·0006		8	−0·0005		7	−0·0027
	9	0·0004		10	−0·0003		10	0·0003		9	0·0014
	11	−0·0002		12	0·0001		12	−0·0002		11	−0·0008
3	3	0·0018	4	4	−0·0029	3	4	−0·0348	2	3	−0·0985
	5	0·0040		6	−0·0006		6	0·0089		5	0·0105
	7	−0·0009		8	0·0002		8	−0·0041		7	−0·0058
	9	0·0004		10	−0·0001		10	0·0022		9	0·0029
	11	−0·0002		12	0·0001		12	−0·0014		11	−0·0017
5	5	−0·0001	6	6	−0·0001	5	6	−0·0019	4	5	−0·0015
	7	0·0001		8	...		8	0·0002		7	−0·0017
	9	...		10	...		10	0·0000		9	0·0009
	11	...		12	...		12	−0·0001		11	−0·0006
									6	7	0·0001
										9	−0·0001
										11	...

TABLE XX (*b*)

VALUES OF COORDINATES. $\beta = 20$.

COEFFICIENTS OF H/h.

n	r	$p_r{}^n\pi_r{}^n$	n	r	$ip_r{}^n\pi_r$	n	r	$p_{-r}{}^n\pi_r{}^n$	n	r	$ip_{-r}{}^n\pi_r{}^n$
1	1	0·7557	2	2	−0·0545	1	2	−0·1154	0	1	−0·2840
	3	0·0008		4	0·0063		4	0·0078		3	−0·0574
	5	0·0092		6	−0·0020		6	−0·0015		5	0·0091
	7	−0·0022		8	0·0008		8	0·0003		7	−0·0055
	9	0·0010		10	−0·0004		10	−0·0001		9	0·0028
	11	−0·0006		12	0·0002		12	−0·0001		11	−0·0018
3	3	0·0151	4	4	−0·0037	3	4	−0·0182	2	3	−0·1088
	5	0·0068		6	−0·0003		6	0·0064		5	0·0132
	7	−0·0014		8	0·0001		8	−0·0032		7	−0·0088
	9	0·0006		10	−0·0001		10	0·0019		9	0·0044
	11	−0·0003		12	0·0001		12	−0·0013		11	−0·0029
5	5	0·0006	6	6	−0·0002	5	6	−0·0023	4	5	−0·0085
	7	0·0004		8	−0·0001		8	0·0005		7	−0·0009
	9	−0·0001		10	...		10	−0·0002		9	0·0005
	11	...		12	...		12	...		11	−0·0005
									6	7	−0·0004
										9	−0·0001
										11	0·0001

Table XX (*c*)

Values of Coordinates. $\beta = 30\cdot63$.

Resonant case. Relative values only.

n	r	$p_r^n\pi_r^n$	n	r	$ip_r^n\pi_r^n$	n	r	$p_{-r}^n\pi_r^n$	n	r	$ip_{-r}^n\pi_r$
1	1	0·5642	2	2	0·3500	1	2	−0·0544	0	1	−0·0130
	3	0·1177		4	−0·0200		4	0·0345		3	−0·0418
	5	0·0004		6	0·0083		6	−0·0113		5	−0·0114
	7	0·0034		8	−0·0030		8	0·0060		7	0·0010
	9	−0·0012		10	0·0014		10	−0·0034		9	−0·0011
	11	0·0007		12	−0·0008		12	0·0023		11	0·0009
3	3	0·1700	4	4	−0·0059	3	4	0·0817	2	3	−0·0730
	5	−0·0060		6	0·0086		6	−0·0220		5	−0·0318
	7	0·0057		8	−0·0029		8	0·0118		7	0·0065
	9	−0·0022		10	0·0014		10	−0·0067		9	−0·0049
	11	0·0013		12	−0·0008		12	0·0045		11	0·0035
5	5	0·0088	6	6	−0·0006	5	6	0·0014	4	5	−0·0558
	7	0·0023		8	0·0005		8	0·0032		7	0·0141
	9	−0·0008		10	...		10	−0·0022		9	−0·0086
	11	0·0005		12	...		12	0·0019		11	0·0055
									6	7	−0·0061
										9	0·0006
										11	...

Table XX (*d*)

Values of Coordinates. $\beta = 40$.

Coefficients of H/h.

n	r	$p_r^n\pi_r^n$	n	r	$ip_r^n\pi_r^n$	n	r	$p_{-r}^n\pi_r^n$	n	r	$ip_{-r}^n\pi_r^n$
1	1	0·1799	2	2	−0·0471	1	2	−0·0481	0	1	−0·1167
	3	−0·0208		4	0·0064		4	0·0011		3	−0·0130
	5	0·0054		6	−0·0021		6	0·0005		5	0·0055
	7	−0·0020		8	0·0009		8	−0·0005		7	−0·0027
	9	0·0009		10	−0·0004		10	0·0004		9	0·0015
	11	−0·0006		12	0·0002		12	−0·0004		11	−0·0011
3	3	−0·0216	4	4	0·0023	3	4	−0·0110	2	3	−0·0201
	5	0·0047		6	−0·0014		6	0·0045		5	0·0092
	7	−0·0017		8	0·0006		8	−0·0025		7	−0·0047
	9	0·0008		10	−0·0003		10	0·0015		9	0·0027
	11	−0·0005		12	0·0002		12	−0·0011		11	−0·0020
5	5	−0·0009	6	6	0·0001	5	6	0·0000	4	5	0·0048
	7	−0·0002		8	−0·0001		8	−0·0005		7	−0·0026
	9	0·0001		10	...		10	0·0004		9	0·0016
	11	−0·0001		12	...		12	−0·0004		11	−0·0011
									6	7	0·0006
										9	−0·0001
										11	...

Table XXI

Values of ϕ.

β	Coefficients of $H/h \,.\, \sin s\theta \sin n\chi \,.\, e^{i\sigma t}$				Coefficients of $H/h \,.\, i \cos s\theta \sin n\chi \,.\, e^{i\sigma t}$			
	s	$n = 1$	3	5	s	$n = 2$	4	6
10·948*	1	0·4907	0·0025	...	0	0·0495	0·0013	...
	3	0·0101	0·0020	0·0001	2	−0·0521	−0·0015	...
	5	0·0034	−0·0023	−0·0001	4	0·0036	0·0000	...
	7	−0·0008	0·0006	...	6	−0·0013	0·0003	...
	9	0·0003	−0·0003	...	8	0·0005	−0·0001	...
	11	−0·0002	0·0001	...	10	−0·0002	0·0001	...
					12	0·0001	...	...
20	1	0·6554	0·0139	0·0006	0	0·0248	0·0016	0·0001
	3	0·0036	0·0003	0·0000	2	−0·0284	−0·0021	−0·0001
	5	0·0090	−0·0039	−0·0002	4	0·0048	0·0004	...
	7	−0·0021	0·0009	0·0001	6	−0·0017	0·0002	...
	9	0·0009	−0·0005	...	8	0·0006	−0·0001	...
	11	−0·0006	0·0003	...	10	−0·0003	0·0001	...
					12	0·0002	...	...
30·63*	1	0·5127	0·1323	0·0071	0	−0·1647	0·0006	0·0001
	3	0·1199	−0·0462	−0·0023	2	0·1752	−0·0039	−0·0003
	5	0·0017	0·0054	−0·0007	4	−0·0153	0·0065	0·0004
	7	0·0033	−0·0038	0·0009	6	0·0070	−0·0042	−0·0003
	9	−0·0010	0·0019	−0·0006	8	−0·0025	0·0017	0·0001
	11	0·0008	−0·0010	0·0002	10	0·0012	−0·0011	...
					12	−0·0008	0·0005	...
40	1	0·1520	−0·0157	−0·0007	0	0·0212	−0·0007	0·0000
	3	−0·0195	0·0082	0·0003	2	−0·0248	0·0013	0·0001
	5	0·0050	−0·0030	0·0000	4	0·0048	−0·0013	−0·0001
	7	−0·0019	0·0012	−0·0001	6	−0·0018	0·0008	0·0001
	9	0·0007	−0·0007	0·0001	8	0·0007	−0·0004	...
	11	−0·0006	0·0004	0·0000	10	−0·0003	0·0003	...
					12	0·0002	−0·0001	...

* Relative values only for resonant cases.

Table XXII

Values of ψ.

β	Coefficients of $H/h \,.\, \sin s\theta \cos n\chi \,.\, e^{i\sigma t}$				Coefficients of $H/h \,.\, i \cos s\theta \cos n\chi \,.\, e^{i\sigma t}$				
	s	$n = 1$	3	5	s	$n = 0$	2	4	6
10·948*	0	...	...	...	1	0·2385	0·0596	0·0013	...
	2	−0·0505	−0·0242	−0·0012	3	0·0486	−0·0643	−0·0012	...
	4	−0·0024	0·0175	0·0010	5	−0·0039	0·0086	−0·0006	−0·0001
	6	0·0006	−0·0064	−0·0004	7	0·0026	−0·0047	0·0010	0·0001
	8	−0·0005	0·0028	0·0001	9	−0·0011	0·0024	−0·0008	...
	10	0·0002	−0·0019	...	11	0·0010	−0·0017	0·0003	...
	12	−0·0003	0·0011	...					
20	0	...	...	...	1	0·3852	0·0659	0·0043	0·0002
	2	−0·1097	−0·0122	−0·0013	3	0·0635	−0·0707	−0·0059	−0·0003
	4	0·0076	0·0098	0·0013	5	−0·0077	0·0109	0·0014	0·0001
	6	−0·0015	−0·0047	−0·0006	7	0·0055	−0·0071	0·0005	0·0001
	8	0·0003	0·0022	0·0002	9	−0·0021	0·0037	−0·0005	−0·0001
	10	−0·0001	−0·0016	−0·0001	11	0·0021	−0·0028	0·0003	...
	12	−0·0001	0·0010	...					
30·63*	0	...	...	...	1	0·0512	0·0614	0·0210	0·0020
	2	−0·0438	0·0569	0·0022	3	0·0558	−0·0401	−0·0374	−0·0039
	4	0·0325	−0·0406	0·0001	5	0·0127	−0·0256	0·0207	0·0025
	6	−0·0100	0·0163	−0·0015	7	−0·0009	0·0050	−0·0084	−0·0008
	8	0·0056	−0·0081	0·0019	9	0·0008	−0·0040	0·0072	0·0001
	10	−0·0026	0·0059	−0·0020	11	−0·0010	0·0034	−0·0031	...
	12	0·0025	−0·0036	0·0008					
40	0	...	...	...	1	0·1500	0·0098	−0·0013	−0·0002
	2	−0·0463	−0·0072	−0·0002	3	0·0128	−0·0140	0·0031	0·0004
	4	0·0012	0·0060	−0·0001	5	−0·0050	0·0076	−0·0026	−0·0003
	6	0·0004	−0·0034	0·0002	7	0·0027	−0·0037	0·0015	0·0001
	8	−0·0005	0·0017	−0·0003	9	−0·0011	0·0022	−0·0014	...
	10	0·0002	−0·0013	0·0004	11	0·0013	−0·0019	0·0006	...
	12	−0·0004	0·0009	−0·0002					

* Relative values only for resonant cases.

TABLE XXIII

VALUES OF u.

β	Coefficients of $\sigma a \cdot H/h \cdot \sin s\theta \sin n\chi \cdot e^{i\sigma t}$				Coefficients of $\sigma a \cdot H/h \cdot i \cos s\theta \sin n\chi \cdot e^{i\sigma t}$			
	s	$n = 2$	4	6	s	$n = 1$	3	5
10·948*	0	...	...	...	1	−0·5962	−0·0693	−0·0045
	2	−0·1342	−0·0071	0·0001	3	−0·0348	0·0726	0·0068
	4	0·0041	−0·0009	−0·0001	5	−0·0167	−0·0144	−0·0027
	6	−0·0078	0·0022	0·0004	7	0·0045	0·0078	0·0007
	8	−0·0007	−0·0026	−0·0002	9	−0·0027	−0·0020	0·0001
	10	−0·0045	0·0020	...	11	0·0018	0·0051	−0·0002
	12	−0·0017	0·0004	...				
20	0	...	...	...	1	−0·8622	−0·0469	−0·0055
	2	−0·2070	−0·0300	−0·0019	3	0·0016	0·0390	0·0083
	4	−0·0001	0·0117	0·0014	5	−0·0475	0·0006	−0·0035
	6	−0·0148	0·0006	−0·0001	7	0·0147	0·0030	0·0008
	8	−0·0012	−0·0014	−0·0004	9	−0·0082	0·0011	−0·0005
	10	−0·0080	0·0015	...	11	0·0067	0·0033	0·0001
	12	−0·0023	0·0005	...				
30·63*	0	...	...	...	1	−0·5443	0·0285	0·0069
	2	−0·5959	−0·1601	−0·0236	3	−0·3038	−0·0424	−0·0010
	4	−0·0240	0·1052	0·0205	5	−0·0174	0·0362	−0·0048
	6	−0·0247	−0·0089	−0·0068	7	−0·0123	−0·0084	0·0004
	8	0·0173	0·0192	0·0008	9	0·0083	−0·0034	−0·0072
	10	0·0014	−0·0132	−0·0003	11	−0·0037	−0·0108	0·0056
	12	0·0097	−0·0056	0·0001				
40	0	...	...	...	1	−0·2427	−0·0042	−0·0012
	2	0·0105	0·0078	−0·0020	3	0·0603	−0·0015	−0·0005
	4	−0·0026	−0·0096	−0·0022	5	−0·0255	0·0021	0·0014
	6	−0·0029	0·0012	0·0009	7	0·0116	−0·0008	−0·0004
	8	−0·0045	−0·0032	−0·0002	9	−0·0069	0·0034	0·0013
	10	−0·0042	−0·0025	...	11	0·0062	0·0013	−0·0012
	12	−0·0029	0·0013					

Table XXIV

Values of v.

β	Coefficients of $\sigma a \,.\, H/h \,.\, \sin s\theta \cos n\chi \,.\, e^{i\sigma t}$					Coefficients of $\sigma a \,.\, H/h \,.\, i \cos s\theta \cos n\chi \,.\, e^{i\sigma t}$			
	s	$n=0$	2	4	6	s	$n=1$	3	5
10.948*	1	0·2386	0·2575	0·0118	0·0004	0	—0·5035	—0·0080	—0·0001
	3	0·1458	—0·2035	—0·0053	—0·0001	2	—0·1265	—0·0493	—0·0025
	5	—0·0194	0·0464	—0·0049	—0·0004	4	—0·0149	0·0812	0·0045
	7	0·0185	—0·0344	0·0072	0·0004	6	0·0050	—0·0410	—0·0025
	9	—0·0095	0·0221	—0·0076	—0·0004	8	—0·0038	0·0237	0·0006
	11	0·0105	—0·0188	0·0041	0·0001	10	0·0026	—0·0202	0·0003
						12	—0·0031	0·0134	—0·0002
	1	0·3852	0·1651	0·0171	0·0013	0	—0·6661	—0·0328	—0·0020
	3	0·1906	—0·2265	—0·0213	—0·0013	2	—0·2408	—0·0067	—0·0011
20	5	—0·0384	0·0590	0·0064	0·0005	4	0·0163	0·0582	0·0063
	7	0·0386	—0·0520	0·0037	0·0005	6	—0·0051	—0·0322	—0·0047
	9	—0·0192	0·0339	—0·0050	—0·0007	8	0·0019	0·0194	0·0021
	11	0·0231	—0·0311	0·0033	0·0001	10	0·0002	—0·0180	—0·0011
						12	—0·0013	0·0122	0·0002
30·63*	1	0·0512	—0·5976	0·0255	0·0037	0	—0·6374	—0·2656	—0·0234
	3	0·1675	—0·0784	—0·1384	—0·0140	2	—0·3371	0·3762	0·0290
	5	0·0637	—0·1474	0·1290	0·0151	4	0·1203	—0·1773	0·0020
	7	—0·0060	0·0432	—0·0671	—0·0065	6	—0·0661	0·1153	—0·0142
	9	0·0072	—0·0378	0·0701	0·0012	8	0·0449	—0·0701	0·0190
	11	—0·0107	0·0401	—0·0376	—0·0001	10	—0·0277	0·0645	—0·0219
						12	0·0300	—0·0432	0·0092
40	1	0·1500	0·0945	—0·0066	—0·0007	0	—0·1358	0·0289	0·0022
	3	0·0383	—0·0565	0·0149	0·0017	2	—0·0600	—0·0507	—0·0030
	5	—0·0249	0·0426	—0·0174	—0·0020	4	—0·0015	0·0371	—0·0002
	7	0·0192	—0·0286	0·0120	0·0010	6	0·0057	—0·0253	0·0017
	9	—0·0097	0·0205	—0·0137	—0·0002	8	—0·0042	0·0154	—0·0032
	11	0·0141	—0·0220	0·0079	...	10	0·0037	—0·0156	0·0042
						12	—0·0049	0·0106	—0·0019

Table XXV (*a*)

Values of $\zeta/He^{i\sigma t}$ for the Diurnal Tide (K_1).

$\beta = 10 \cdot 948.$

s	Coefficients* of $\sin s\theta \sin n\chi$ $(s > 0)$, $\theta \sin n\chi$ $(s = 0)$: $n = 1$	3	5	7	Coefficients* of $i\cos s\theta \sin n\chi$: $n = 2$	4	6
0	0·301	0·336	0·016		—1·085	0·020	0·001
2	0·188	—0·304	—0·011		—0·184	—0·010	
4	—0·199	0·085	0·000		0·003	—0·001	
6	0·036	—0·021	0·003		—0·004	0·001	
8	—0·019	0·012	—0·002		0·000	—0·001	
10	0·011	—0·007	0·002		—0·001	0·001	
12	—0·005	0·003	0·000				
1	—1·632	—0·190	—0·012		1·250	—0·042	—0·003
3	—0·032	0·066	0·006		—0·009	0·056	0·003
5	—0·009	—0·008	—0·001		0·039	—0·032	—0·002
7	0·002	0·003			—0·014	0·012	0·001
9	—0·001	—0·001			0·008	—0·007	
11		0·001			—0·005	0·004	
13					0·002	—0·001	

* Relative values only ; resonant case.

Table XXV (*b*)

Values of $\zeta/He^{i\sigma t}$ for the Diurnal Tide (K_1).

$\beta = 20.$

s	Coefficients of $\sin s\theta \sin n\chi$ $(s > 0)$, $\theta \sin n\chi$ $(s = 0)$: $n = 1$	3	5	7	Coefficients of $i\cos s\theta \sin n\chi$: $n = 2$	4	6
0	1·513	0·370	0·040	0·003	—1·730	—0·099	—0·006
2	1·003	—0·441	—0·045	—0·003	—1·017	—0·075	—0·005
4	—0·465	0·161	0·016	0·001	0·000	0·015	0·002
6	0·110	—0·045	—0·001		—0·012	0·001	
8	—0·063	0·030	—0·002		—0·001	—0·001	
10	0·037	—0·018	0·002		—0·004	0·001	
12	—0·016	0·007	—0·001		—0·001	0·000	
1	—4·311	—0·235	—0·027		2·581	0·140	0·007
3	0·003	0·065	0·014		0·160	0·048	0·006
5	—0·048	0·001	—0·004		0·034	—0·040	—0·005
7	0·011	0·002	0·001		—0·016	0·016	0·002
9	—0·005	0·001			0·010	—0·009	—0·001
11	0·003	0·001			—0·007	0·007	
13					0·003	—0·002	

TABLE XXV (*c*)

VALUES OF $\zeta/He^{i\sigma t}$ FOR THE DIURNAL TIDE (K_1).

$\beta = 30\cdot63.$

s	Coefficients* of sin $s\theta$ sin $n\chi$ ($s>0$), θ sin $n\chi$ ($s=0$)				Coefficients* of $i$cos $s\theta$ sin $n\chi$		
	$n=1$	3	5	7	$n=2$	4	6
0	2·680	−2·385	0·083	0·014	1·185	−2·251	−0·194
2	−0·549	1·308	−0·280	−0·034	−2·281	−0·613	−0·090
4	−0·133	−0·322	0·228	0·028	−0·046	0·201	0·039
6	−0·211	0·247	−0·111	−0·014	−0·031	−0·011	−0·009
8	0·051	−0·104	0·062	0·004	0·017	0·018	0·001
10	−0·043	0·071	−0·041		0·001	−0·010	
12	0·020	−0·025	0·012		0·006	−0·004	
1	−4·168	0·218	0·053		0·116	3·184	0·290
3	−0·776	−0·108	−0·002		1·290	−0·672	−0·034
5	−0·027	0·055	−0·007		−0·367	0·236	−0·012
7	−0·013	−0·009	0·000		0·162	−0·120	0·018
9	0·007	−0·003	−0·006		−0·088	0·075	−0·017
11	−0·003	−0·008	0·004		0·058	−0·048	0·011
13					−0·022	0·015	−0·003

* Relative values only; resonant case.

TABLE XXV (*d*)

VALUES OF $\zeta/He^{i\sigma t}$ FOR THE DIURNAL TIDE (K_1).

$\beta = 40.$

s	Coefficients of sin $s\theta$ sin $n\chi$ ($s>0$), θ sin $n\chi$ ($s=0$)				Coefficients of $i$cos $s\theta$ sin $n\chi$.		
	$n=1$	3	5	7	$n=2$	4	6
0	1·028	0·506	−0·030	−0·003	−1·126	0·394	0·046
2	0·719	−0·431	0·048	0·006	−0·447	0·039	−0·010
4	−0·282	0·164	−0·036	−0·005	−0·007	−0·024	−0·006
6	0·133	−0·084	0·022	0·002	−0·005	0·002	0·002
8	−0·067	0·047	−0·015	−0·001	−0·006	−0·004	
10	0·045	−0·032	0·011		−0·004	−0·003	
12	−0·021	0·012	−0·003		−0·002	0·001	
1	−2·427	−0·042	−0·012		1·600	−0·505	−0·037
3	0·201	−0·005	−0·002		−0·049	0·142	0·005
5	−0·051	0·004	0·003		0·070	−0·064	0·002
7	0·017	−0·001	−0·001		−0·036	0·033	−0·004
9	−0·008	0·004	0·001		0·022	−0·021	0·004
11	0·006	0·001	−0·001		−0·016	0·015	−0·003
13					0·006	−0·005	0·001

TABLE XXVI (*a*)

VALUES OF ζ_1, ζ_2, R, γ FOR THE DIURNAL TIDE (K_1).

$\beta = 10\cdot948$.

	χ	$\theta = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
ζ_1*	0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·104	−0·204	−0·285	−0·342	−0·376	−0·380	−0·352	−0·285	−0·159
	40	0·000	−0·197	−0·366	−0·492	−0·570	−0·603	−0·606	−0·594	−0·568	−0·510
	60	0·000	−0·260	−0·466	−0·602	−0·661	−0·667	−0·669	−0·693	−0·792	−0·992
	80	0·000	−0·288	−0·510	−0·643	−0·676	−0·660	−0·664	−0·707	−0·908	−1·342
	90	0·000	−0·291	−0·515	−0·647	−0·676	−0·657	−0·661	−0·706	−0·922	−1·392
ζ_2*	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	0·009	0·034	0·077	0·138	0·208	0·287	0·374	0·460	0·548
	40	0·000	0·014	0·051	0·115	0·191	0·273	0·370	0·481	0·635	0·871
	60	0·000	0·012	0·047	0·100	0·152	0·210	0·268	0·346	0·498	0·796
	80	0·000	0·005	0·020	0·040	0·056	0·071	0·094	0·120	0·183	0·321
	90	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
R*	0	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	20	0·00	0·10	0·21	0·30	0·37	0·43	0·48	0·51	0·54	0·57
	40	0·00	0·20	0·37	0·51	0·60	0·66	0·71	0·76	0·85	1·01
	60	0·00	0·26	0·47	0·61	0·68	0·70	0·72	0·78	0·94	1·27
	80	0·00	0·29	0·51	0·64	0·68	0·67	0·67	0·72	0·93	1·38
	90	0·00	0·29	0·52	0·65	0·68	0·66	0·66	0·71	0·92	1·39
γ	0	180°	175°	170°	164°	157°	149°	141°	131°	121°	101°
	20	180	175	171	165	158	151	143	133	122	106
	40	180	176	172	168	161	156	149	141	132	120
	60	180	177	174	171	167	163	158	153	148	141
	80	180	179	178	176	175	174	172	170	169	167
	90	180	180	180	180	180	180	180	180	180	180

* Relative values ; resonant case. The values of γ change by 180° as R becomes infinite.

TABLE XXVI (*b*)

VALUES OF ζ_1, ζ_2, R, γ FOR THE DIURNAL TIDE (K_1).

$\beta = 20$.

	χ	$\theta = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
ζ_1/H	0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·147	−0·289	−0·412	−0·504	−0·566	−0·602	−0·606	−0·558	−0·423
	40	0·000	−0·271	−0·506	−0·674	−0·762	−0·803	−0·834	−0·867	−0·950	−1·056
	60	0·000	−0·357	−0·626	−0·772	−0·786	−0·755	−0·765	−0·830	−1·134	−1·746
	80	0·000	−0·400	−0·673	−0·789	−0·738	−0·658	−0·667	−0·749	−1·204	−2·207
	90	0·000	−0·405	−0·678	−0·790	−0·729	−0·642	−0·653	−0·737	−1·213	−2·272
ζ_2/H	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·004	−0·006	0·007	0·034	0·070	0·121	0·196	0·304	0·460
	40	0·000	−0·007	−0·009	0·011	0·037	0·062	0·112	0·192	0·355	0·693
	60	0·000	−0·006	−0·008	0·013	0·023	0·022	0·050	0·092	0·225	0·595
	80	0·000	−0·002	−0·003	0·007	0·008	0·003	0·013	0·024	0·072	0·230
	90	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
R/H	0	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	20	0·00	0·15	0·29	0·41	0·51	0·57	0·62	0·64	0·64	0·63
	40	0·00	0·27	0·51	0·67	0·76	0·81	0·85	0·89	1·02	1·26
	60	0·00	0·36	0·63	0·77	0·79	0·76	0·77	0·84	1·16	1·84
	80	0·00	0·40	0·67	0·79	0·74	0·66	0·67	0·75	1·21	2·22
	90	0·00	0·41	0·68	0·79	0·73	0·64	0·65	0·74	1·21	2·27
γ	0	180°	182°	181°	179°	176°	172°	167°	160°	148°	126°
	20	180	182	181	179	176	173	169	162	151	133
	40	180	181	181	179	177	176	172	167	159	147
	60	180	181	181	179	178	178	176	174	169	161
	80	180	180	180	179	179	180	179	178	177	174
	90	180	180	180	180	180	180	180	180	180	180

TABLE XXVI (c)

VALUES OF ζ_1, ζ_2, R, γ FOR THE DIURNAL TIDE (K_1).

$\beta = 30 \cdot 63$.

	χ	$\theta = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
ζ_1*	0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·384	−0·709	−0·985	−1·219	−1·398	−1·550	−1·722	−1·996	−2·444
	40	0·000	−0·719	−1·276	−1·612	−1·704	−1·643	−1·527	−1·401	−1·607	−2·458
	60	0·000	−0·956	−1·607	−1·795	−1·604	−1·216	−0·769	−0·320	0·138	0·578
	80	0·000	−1·079	−1·727	−1·750	−1·413	−0·871	−0·227	0·288	1·395	3·804
	90	0·000	−1·095	−1·739	−1·735	−1·383	−0·823	−0·149	0·365	1·567	4·311
ζ_2*	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·022	−0·087	−0·227	−0·448	−0·682	−0·883	−1·053	−1·095	−0·804
	40	0·000	−0·035	−0·157	−0·385	−0·626	−0·835	−1·105	−1·447	−2·046	−2·988
	60	0·000	−0·032	−0·161	−0·384	−0·490	−0·545	−0·797	−1·082	−1·982	−4·222
	80	0·000	−0·013	−0·070	−0·165	−0·178	−0·171	−0·286	−0·373	−0·788	−2·041
	90	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
R*	0	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	20	0·00	0·39	0·71	1·01	1·30	1·56	1·78	2·01	2·26	2·57
	40	0·00	0·72	1·28	1·65	1·82	1·84	1·89	2·00	2·60	3·85
	60	0·00	0·96	1·61	1·93	1·67	1·34	1·11	1·13	1·98	4·26
	80	0·00	1·08	1·73	1·76	1·42	0·89	0·36	0·47	1·60	4·31
	90	0·00	1·10	1·74	1·74	1·38	0·82	0·15	0·37	1·57	4·31
γ	0	180°	183°	187°	193°	200°	205°	208°	207°	202°	187°
	20	180	183	187	193	200	206	210	211	209	198
	40	180	183	187	193	200	207	216	226	232	231
	60	180	182	186	192	197	204	226	253	274	278
	80	180	181	182	185	188	191	232	308	331	332
	90	180	180	180	180	180	180	180	0	0	0

* Relative values ; resonant case. The values of γ change by 180° as R becomes infinite.

TABLE XXVI (*d*)

VALUES OF ζ_1, ζ_2, R, γ FOR THE DIURNAL TIDE (K_1).

$\beta = 40$.

	χ	$\theta = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
ζ_1/H	0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·014	−0·036	−0·050	−0·050	−0·041	−0·024	0·014	0·094	0·235
	40	0·000	−0·019	−0·046	−0·071	−0·083	−0·090	−0·102	−0·116	−0·110	−0·018
	60	0·000	−0·023	−0·049	−0·085	−0·094	−0·114	−0·175	−0·236	−0·456	−0·908
	80	0·000	−0·023	−0·053	−0·103	−0·099	−0·128	−0·233	−0·283	−0·688	−1·773
	90	0·000	−0·023	−0·054	−0·107	−0·099	−0·131	−0·243	−0·289	−0·720	−1·907
ζ_2/H	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
	20	0·000	−0·017	−0·042	−0·035	−0·020	−0·012	0·008	0·048	0·074	0·075
	40	0·000	−0·015	−0·031	−0·004	0·022	0·036	0·080	0·151	0·300	0·604
	60	0·000	−0·006	−0·012	0·016	0·018	0·013	0·061	0·100	0·292	0·878
	80	0·000	−0·001	−0·005	0·004	−0·006	−0·017	0·004	0·005	0·086	0·404
	90	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
R/H	0	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
	20	0·00	0·02	0·06	0·06	0·06	0·04	0·02	0·05	0·12	0·25
	40	0·00	0·02	0·06	0·07	0·08	0·10	0·13	0·19	0·32	0·61
	60	0·00	0·02	0·05	0·09	0·10	0·12	0·19	0·26	0·54	1·26
	80	0·00	0·02	0·05	0·10	0·10	0·13	0·23	0·28	0·69	1·82
	90	0·00	0·02	0·05	0·11	0·10	0·13	0·24	0·29	0·72	1·91
γ	0	180°	231°	233°	222°	222°	230°	264°	8°	0°	344°
	20	180	230	229	215	202	196	172	74	38	18
	40	180	218	214	183	165	158	142	128	110	92
	60	180	195	194	169	169	173	161	157	147	136
	80	180	182	185	178	183	188	179	179	173	167
	90	180	180	180	180	180	180	180	180	180	180

APPENDIX

Fourier Expansions of Associated Legendre Functions

By A. T. Doodson, F.R.S.

1—Introduction

The lack of Tables of the Associated Legendre Functions in a suitable form has been the source of much inconvenience to research workers. It was intended, in connexion with the investigations on "Tides in Oceans bounded by Meridians", to publish a Table of these functions, $P_r^n(\cos\theta)$, which was computed for r up to 12, n up to 6, and θ at intervals of 10°, but in the course of the work it became necessary to multiply the functions by trigonometrical expressions and then to integrate the products, with respect to θ. Operations involving integration of these functions, however, are not readily expressed mathematically, and the numerical application of the results would be very cumbrous, even if possible.

These difficulties were overcome by expressing the functions in their Fourier Expansions, which are finite series for integral values of r and n. Such series can be combined with trigonometrical expressions and integration with regard to θ is obviously easy.

Further advantages of the Fourier expansions, even if integration processes are not required, can be claimed. Normally, the solution of a problem is expressed as a series of a very large number of these functions, and it is not a simple matter to deduce, by inspection, the character of the result. If, however, the Fourier expansions are used, then the solution is ultimately expressed in a single Fourier expansion, and this, being in familiar trigonometrical terms, can be more readily interpreted. Questions of convergence also become simplified. Again, the flexibility of the trigonometrical expansion is such that any values of θ can be adopted for numerical representation, and we are not limited to specified values of θ in tables of functions. To illustrate a solution for a number of values of θ is less laborious by this method than by the use of tables of the functions. Hence it may be claimed that the Fourier expansions are much more serviceable than tables of the functions.

2—Recurrence Formulae

By Ferrer's definition, $P_r^n(\cos\theta)$ is the product of $\sin^n\theta$ and a descending series in powers of $\cos\theta$, whose first term is $(\cos\theta)^{r-n}$. It is readily deduced that we have finite series in the forms

$$P_r^n(\cos\theta) = \Sigma a_s^n \cos s\theta \qquad (n \text{ even}, s \leq r), \quad . \; . \; . \; . \; . \quad (2.1)$$

$$P_r^n(\cos\theta) = \Sigma b_s^n \sin s\theta \qquad (n \text{ odd}, s \leq r), \quad . \; . \; . \; . \; . \quad (2.2)$$

and substitutions in the differential equation

$$\sin\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\,\mathrm{P}_r^n\right)+\{r\,(r+1)\sin^2\theta-n^2\}\,\mathrm{P}_r^n=0\quad . \; . \; . \quad (2.3)$$

yield the general recurrence equations

$$\{r\,(r+1)-(s-1)\,(s-2)\}\,a_{s-2}^n=\{2r\,(r+1)-2s^2-4n^2\}\,a_s^n+\{(s+1)\,(s+2)-r\,(r+1)\}\,a_{s+2}^n, \quad (2.4)$$

$$\{r\,(r+1)-(s-1)\,(s-2)\}\,b_{s-2}^n=\{2r\,(r+1)-2s^2-4n^2\}\,b_s^n+\{(s+1)\,(s+2)-r\,(r+1)\}\,b_{s+2}^n, \quad (2.5)$$

with the special relations

$$2r\,(r+1)\,a_0^n=\{2\,r\,(r+1)-2.2^2-4n^2\}\,a_2^n+\{4.3-r\,(r+1)\}a_4^n, \quad . \quad (2.6)$$

$$\{r\,(r+1)-2.1^2-4n^2\}\,a_1^n=-\{3.2-r\,(r+1)\}\,a_3^n, \quad . \; . \quad (2.7)$$

$$\{3\,r\,(r+1)-2.1^2-4n^2\}\,b_1^n=-\{3.2-r\,(r+1)\}\,b_3^n. \quad . \; . \quad (2.8)$$

By taking the last term in the series ($s = r$) to have its coefficient as unity, for the purposes of the first stage of the computation, the relative values of the coefficients were readily determined by these recurrence formulae.

3—Computation of Last Terms of Series

Several methods of computation are available, but the following is simple.

Let c_r^n be the last term of a series ($s = r$), whether with coefficients a_s^n or b_s^n. Then we use two well-known formulae

$$(2\,r+1)\cos\theta\,\mathrm{P}_r^n=(r-n+1)\,\mathrm{P}_{r+1}^n+(r+n)\,\mathrm{P}_{r-1}^n, \quad . \; . \; . \quad (3.1)$$

$$\mathrm{P}_{r+1}^n-\mathrm{P}_{r-1}^n=(2\,r+1)\sin\theta\,\mathrm{P}_r^{n-1}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.2)$$

From the former we deduce

$$\frac{2\,r+1}{2}\,c_r^n=(r-n+1)\,c_{r+1}^n, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.3)$$

and from the latter, with $r = n$,

$$c_n^n=\tfrac{1}{2}\,(2n-1)\,c_{n-1}^{n-1} \qquad (n \text{ odd}), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.4)$$

$$c_n^n=-\tfrac{1}{2}\,(2n-1)\,c_{n-1}^{n-1} \qquad (n \text{ even}), \quad . \; . \; . \; . \; . \; . \; . \quad (3.5)$$

with

$$c_0^0=1, \qquad c_1^1=1. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.6)$$

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Hence we get

$$c_r^n = \pm \frac{(2r)!}{2^{2r-1} r! \, (r-n)!}, \quad \ldots \ldots \ldots \quad (3.7)$$

and the positive sign is taken when $\frac{1}{2}n$ and n are even, and also when $\frac{1}{2}(n+1)$ is even, with n odd.

In the calculation of c_r^n, however, the relations (3.3) to (3.6) were used to get successive values.

4—Use of the Integral of the Square of the Function

These functions increase rapidly in value with r and n, and in order to use numerical values less than unity many investigators have found it convenient to work with such expressions as

$$P_r^n (\cos \theta)/P_r^n (0) \quad \text{and} \quad P_r^n (\cos \theta)/P_r^{n-1} (0).$$

The customary methods of determining coefficients of these functions, however, make use of the integrals

$$\int_{-1}^{1} P_r^n P_s^n \, d(\cos \theta) = 0 \qquad (r \neq s),$$

and

$$\int_{-1}^{1} \{P_r^n\}^2 \, d(\cos \theta) = \frac{2}{2r+1} \frac{(r+n)!}{(r-n)!} = \{N_r^n\}^2, \text{ say} \quad \ldots \quad (4.1)$$

in a manner analogous to that used in the determination of ordinary Fourier coefficients.

For this reason, it was decided to compute the functions P_r^n/N_r^n as being the most useful form.

5—Tables of Expansions

It is convenient to use the notation

$$F_r^n (\theta) = P_r^n (\cos \theta)/N_r^n,$$

and the expansions of $F_r^n(\theta)$ are given for r and n up to 12 in four tables herewith. The values of N_r^n are also given in the same tables and an additional table gives a useful list of values of $P_r^n (0)$.

These tables have been compared with direct calculations of $P_r^n (0)$ and $\frac{\partial}{\partial (\cos \theta)} P_r^n (\cos \theta)$ at $\theta = \frac{\pi}{2}$, and other tests have been made using the series in $\cos \theta$.

6—Remarks on the Tables

It will be noted that the maximum coefficients in the expansions for $F_r^n(\theta)$ are about 0.5 to 1.0 in each case. This is a very useful feature, as a great drawback to tables of P_r^n is that the values become very large.

The values of N_r^n are also conveniently expressed with factors 10^n.

In connexion with matters of convergence it is useful to consider the values of $P_r^n(0)/N_r^n$; that is, the maximum values of $F_r^n(\theta)$.

We have

$$P_r^n(0) = \left(-\tfrac{1}{2}\right)^{\frac{1}{2}(r-n)} \frac{1\cdot 3\;.\;.\;.\;(r+n-1)}{\{\frac{1}{2}(r-n)\}\,!},$$

so that $P_r^n(0)/N_r^n$ becomes, after some reduction,

$$\left(-\tfrac{1}{2}\right)^{\frac{1}{2}(r-n)} \left(\tfrac{1}{2}\right)^{\frac{1}{2}(r+n)} \frac{\{(r+n)\,!\,(r-n)\,!\}^{\frac{1}{2}} \left\{\dfrac{2r+1}{2}\right\}^{\frac{1}{2}}}{\{\frac{1}{2}(r+n)\}\,!\,\{\frac{1}{2}(r-n)\}\,!}.$$

The asymptotic form of $x\,!$, when x is large, is $(2\pi x)^{\frac{1}{2}} x^x e^{-x}$; whence we get the expression

$$(-1)^{\frac{1}{2}(r-n)} \frac{\sqrt{(2r+1)/\pi}}{(r+n)^{\frac{1}{4}}(r-n)^{\frac{1}{4}}}$$

when $(r-n)$ is large.

Hence, when n is small, we get $(-1)^{\frac{1}{2}(r-n)}\sqrt{\dfrac{2}{\pi}}$, which is independent of r as regards magnitude.

When n is equal to r, we must return to the original expression, which gives, after a similar reduction, $\left(\dfrac{r}{\pi}\right)^{\frac{1}{4}}$. This varies so slowly that it is hardly likely to need consideration in matters of convergence.

Hence we conclude that we need only consider the coefficients of $F_r^n(\theta)$ with regard to convergence, taking the absolute maximum value of $F_r^n(\theta)$ itself as being approximately constant for large values of r.

Table A—I

Values of $F_r^n(\theta) = P_r^n(\cos\theta)/N_r^n$.

Coefficients of $\cos s\theta$.

n	r	$s = 0$	2	4	6	8	10	12	$N_r^n/10^n$
0	0	0·7071068							1·4142136
	2	0·3952847	1·1858541						0·6324555
	4	0·2983107	0·6629126	1·1600971					0·4714045
	6	0·2489756	0·5228487	0·6274184	1·1502671				0·3922323
	8	0·2179845	0·4484253	0·4932678	0·6107125	1·1450859			0·3429972
	10	0·1962437	0·3997558	0·4242306	0·4772594	0·6009934	1·1418874		0·3086067
	12	0·1799198	0·3645129	0·3797010	0·4098359	0·4672130	0·5946347	1·1397165	0·2828427
2	2	0·4841229	−0·4841229						0·03098387
	4	0·3144471	0·4192627	−0·7337098					0·08944272
	6	0·2551240	0·4337109	0·1530744	−0·8419093				0·16076739
	8	0·2210766	0·4042544	0·2779249	0·0000000	−0·9032560			0·2435038
	10	0·1980525	0·3740991	0·3035907	0·1663910	−0·0992508	−0·9428825		0·3363671
	12	0·1810844	0·3480583	0·3037671	0·2221093	0·0844015	−0·1688030	−0·9706175	0·4383971
4	4	0·4159743	−0·5546324	0·1386581					0·009465728
	6	0·2794744	0·1397372	−0·7266334	0·3074218				0·05283356
	8	0·2318671	0·2649910	−0·2384919	−0·6889766	0·4306104			0·15323339
	10	0·2040252	0·2947031	−0·0129540	−0·4031927	−0·6055987	0·5230171		0·3395809
	12	0·1848185	0·2976297	0·0956096	−0·1904190	−0·4651665	−0·5168517	0·5943794	0·6443095
6	6	0·3784101	−0·5676151	0·2270461	−0·0378410				0·008584437
	8	0·2579979	0·0000000	−0·6191950	0·4717676	−0·1105705			0·07161100
	10	0·2164014	0·1522825	−0·3801338	−0·4070408	0·6034051	−0·1849145		0·2881440
	12	0·1920125	0·2094682	−0·1832847	−0·4471980	−0·1800429	0·6633160	−0·2542711	0·8434304
8	8	0·3532782	−0·5652451	0·2826226	−0·0807493	0·0100937			0·015689183
	10	0·2428386	−0·0809462	−0·5088047	0·5377141	−0·2274203	0·0366185		0·17460668
	12	0·2050133	0·0745503	−0·4193454	−0·1686256	0·5972898	−0·3603264	0·0714440	0·9005372
10	10	0·3347299	−0·5578832	0·3187904	−0·1195464	0·0265659	−0·0026566		0·04813574
	12	0·2312862	−0·1321635	−0·4130110	0·5506814	−0·3193952	0·0941165	−0·0115142	0·6705224
12	12	0·3202012	−0·5489164	0·3430727	−0·1524768	0·0457430	−0·0083169	0·0006931	0·2227911

Table A—II

Values of $F_r^n(\theta) = P_r^n(\cos\theta)/N_r^n$.

Coefficients of $\cos s\theta$.

n	r	$s = 1$	3	5	7	9	11	$N_r^n/10^n$
0	1	1·2247449						0·8164966
	3	0·7015607	1·1692679					0·5345225
	5	0·5496581	0·6412678	1·1542820				0·4264014
	7	0·4680247	0·5054666	0·6177925	1·1473290			0·3651484
	9	0·4148112	0·4345641	0·4842285	0·6052857	1·1433174		0·3244428
	11	0·3765230	0·3884761	0·4162244	0·4717210	0·5975133	1·1407072	0·2948839
2	3	0·6404344	−0·6404344					0·05855401
	5	0·5310202	0·2655101	−0·7965303				0·12358287
	7	0·4595911	0·3492892	0·0674067	−0·8762870			0·2007984
	9	0·4101763	0·3515797	0·2176446	−0·0544111	−0·9249893		0·2887359
	11	0·3736597	0·3380730	0·2605450	0·1224350	−0·1368391	−0·9578736	0·3862867
4	5	0·4598770	−0·6898154	0·2299385				0·02568622
	7	0·4311346	−0·0862269	−0·7185577	0·3736500			0·09418280
	9	0·3952557	0·1129302	−0·3387906	−0·6493487	0·4799534		0·2337156
	11	0·3646545	0·1910095	−0·1128693	−0·4427948	−0·5608734	0·5608734	0·4749913
6	7	0·3663940	−0·6595092	0·3663940	−0·0732788			0·02881440
	9	0·3659357	−0·2439571	−0·5227653	0·5489036	−0·1481168		0·15146503
	11	0·3479947	−0·0386661	−0·4391362	−0·2899956	0·6401998	−0·2203967	0·5076854
8	9	0·3079808	−0·6159616	0·4399726	−0·1539904	0·0219986		0·06118887
	11	0·3197851	−0·3197851	−0·3426269	0·5862726	−0·2969433	0·0532975	0·4198777
10	11	0·2675514	−0·5733244	0·4777703	−0·2229595	0·0573324	−0·0063703	0·2107769

Table A—III

Values of $F_r^n(\theta) = P_r^n(\cos\theta)/N_r^n$.

Coefficients of $\sin s\theta$.

n	r	$s = 2$	4	6	8	10	12	$N_r^n/10^n$
1	2	0·9682459						0·15491933
	4	0·2964635	1·0376224					0·21081851
	6	0·1613546	0·3872511	1·0649404				0·2541956
	8	0·1056949	0·2325287	0·4318390	1·0795974			0·2910428
	10	0·0762304	0·1617952	0·2730294	0·4584197	1·0887469		0·3236694
	12	0·0583688	0·1216016	0·1968788	0·2992558	0·4760888	1·0950042	0·3532704
3	4	0·7843688	−0·3921844					0·03346640
	6	0·4592233	0·6122977	−0·5612729				0·09646043
	8	0·3078917	0·5131529	0·4447325	−0·6670987			0·19782345
	10	0·2244163	0·4030334	0·4740222	0·3114349	−0·7396579		0·3430285
	12	0·1728173	0·3216322	0·4171084	0·4134813	0·2067407	−0·7925059	0·5369246
5	6	0·6554255	−0·5243404	0·1310851				0·02478113
	8	0·4777193	0·2866316	−0·6688070	0·2388597			0·11049817
	10	0·3584358	0·4096409	−0·0153615	−0·6963896	0·3307851		0·3221547
	12	0·2799137	0·3965444	0·2332614	−0·2127344	−0·6736589	0·4077409	0·7513875
7	8	0·5652451	−0·5652451	0·2422479	−0·0403747			0·03922296
	10	0·4626461	0·0755341	−0·6089934	0·4154373	−0·0896967		0·2376096
	12	0·3727514	0·2795636	−0·2573760	−0·4934519	0·5342770	−0·1428880	0·9005372
9	10	0·4989859	−0·5702696	0·3207767	−0·0950449	0·0118806		0·10763479
	12	0·4392415	−0·0610058	−0·5016030	0·5043144	−0·2033526	0·0311807	0·8253560
11	12	0·4481883	−0·5602354	0·3734903	−0·1493961	0·0339537	−0·0033954	0·4547704

TABLE A—IV

VALUES OF $F_r^n(\theta) = P_r^n(\cos\theta)/N_r^n$.

COEFFICIENTS OF $\sin s\theta$.

n	r	$s = 1$	3	5	7	9	11	$N_r^n/10^n$
1	1	0·8660254						0·11547005
	3	0·2025231	1·0126157					0·18516402
	5	0·1003534	0·3512368	1·0537105				0·23354968
	7	0·0625424	0·2026374	0·4127800	1·0732280			0·2732520
	9	0·0437249	0·1374212	0·2552108	0·4466190	1·0846461		0·3077935
	11	0·0327721	0·1014375	0·1811384	0·2874063	0·4680617	1·0921440	0·3387958
3	3	0·7843688	−0·2614563					0·01434274
	5	0·3251821	0·7045612	−0·4877732				0·06054300
	7	0·1949880	0·5069687	0·5243010	−0·6196284			0·14198592
	9	0·1342617	0·3708180	0·4986863	0·3740147	−0·7064723		0·2646308
	11	0·0998648	0·2837427	0·4161031	0·4446359	0·2560025	−0·7680075	0·4336057
5	5	0·7271294	−0·3635647	0·0727129				0·008122695
	7	0·3592789	0·4742481	−0·6179596	0·1868250			0·05650968
	9	0·2362105	0·4724210	0·1214797	−0·6917593	0·2868270		0·19554051
	11	0·1722831	0·4019938	0·3240562	−0·1255205	−0·6889682	0·3709829	0·5026835
7	7	0·6854604	−0·4112762	0·1370921	−0·0195846			0·010781360
	9	0·3697279	0·3169096	−0·6187283	0·3357733	−0·0641365		0·10493805
	11	0·2567731	0·4035005	−0·1179060	−0·5615819	0·4812311	−0·1161592	0·4816327
9	9	0·6533259	−0·4355506	0·1866646	−0·0466661	0·0051851		0·02596024
	11	0·3715567	0·2064204	−0·5750282	0·4275851	−0·1484261	0·0206420	0·3252358
11	11	0·6274636	−0·4481883	0·2240942	−0·0746981	0·0149396	−0·0013581	0·09886314

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TABLE A—V

TABLE OF $P_r^n(0)/10^n$.

n	r		n	r	
0	0	1·00	1	1	0·10
	2	−0·50		3	−0·15
	4	0·375		5	0·1875
	6	−0·3125		7	−0·21875
	8	0·2734375		9	0·24609375
	10	−0·24609375		11	−0·27070313
	12	0·22558594			
2	2	0·03	3	3	0·015
	4	−0·075		5	−0·0525
	6	0·13125		7	0·118125
	8	−0·196875		9	−0·2165625
	10	0·27070313		11	0·35191406
	12	−0·35191406			
4	4	0·0105	5	5	0·00945
	6	−0·04725		7	−0·051975
	8	0·1299375		9	0·16891875
	10	−0·28153125		11	−0·42229688
	12	0·52787109			
6	6	0·010395	7	7	0·0135135
	8	−0·0675675		9	−0·10135125
	10	0·25337813		11	0·43074281
	12	−0·71790469			
8	8	0·02027025	9	9	0·03445943
	10	−0·17229713		11	−0·32736454
	12	0·81841134			
10	10	0·06547291	11	11	0·137493106
	12	−0·68746553			
12	12	0·31623414			

The values are exact where the number of figures after the decimal point is less than eight.